Inflation of digitally convex polyominoes

Jean-Pierre Borel, Lama Tarsissi , Laurent Vuillon

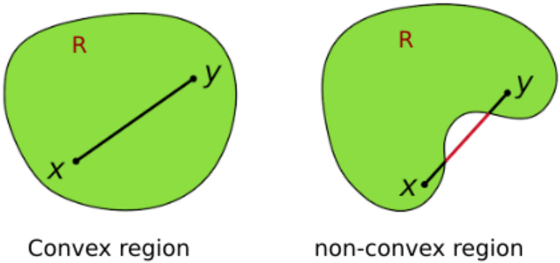
Journées Montoises, LABRI,

14-09-2018

Outline

1. Transformations of digitally convex polyominoes
2. Intuitive approach
3. Another approach with combinatorics on words

The continuous version of convexity



Definition

A set in Euclidean geometry is convex if and only if for any pair of points p_1 , p_2 in a region R, the line segment joining them is completely included in R.

What about the discrete version?

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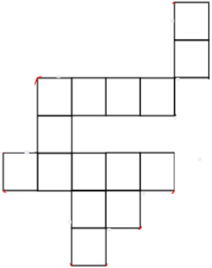

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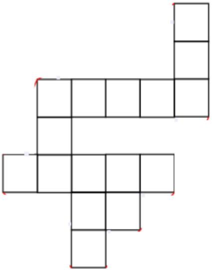

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A polyomino is a finite 4-connected set of unit squares in the lattice $\mathbb{Z}^2$.

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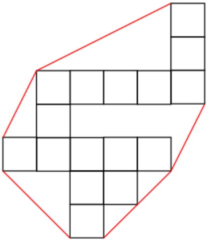

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A polyomino P is a digitally convex polyomino (DC) if its convex hull contains no integer point outside P.

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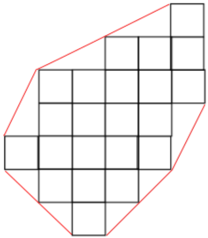

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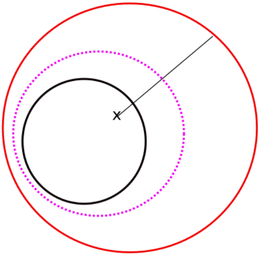
2. Deflate DC polyominoes

Inflate and deflate of convexes

It can be done in the slowest way by passing from:

1. An empty set to a convex
2. a small C_1 into a bigger C_2



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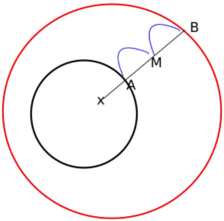

$$MA = \lambda\ AB; \lambda \in [0, 1]$$

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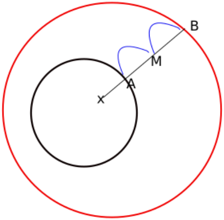

How can we do it in the digital case?

As in the continuous case, we have the expansion from an interior pixel by adding

pixel by pixel.

Deflate of a DC polyomino P

The corners of P are the pixels such that a vertex is an angle of the convex hull.

To deflate P, we do it step by step by removing one unit square at each time.

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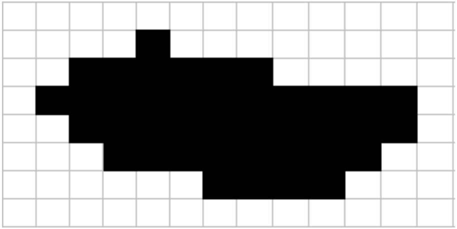

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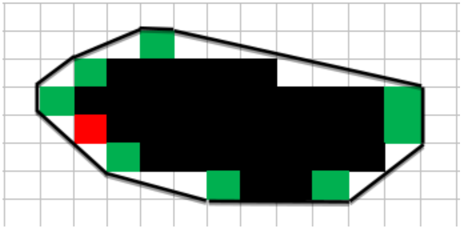

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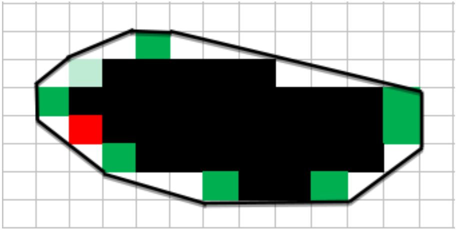

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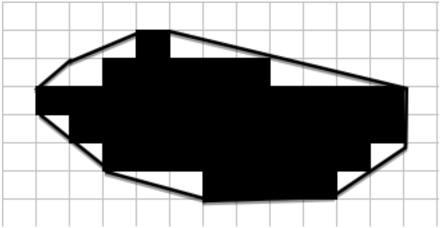

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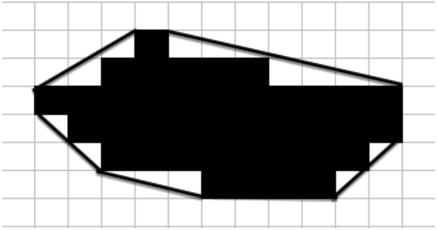


Deflate of a DC polyomino P

The corners of P are the pixels such that a vertex is an angle of the convex hull.

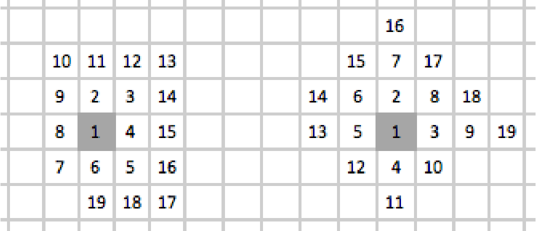
To deflate P, we do it step by step by removing one unit square at each time.

However, it does not give a practical way to choose the unit square that we must add at each step.

Outline

3. Inflate DC polyominoes

The spiral construction



1. By adding a corner around the polyomino from a unit square polyomino
2. by adding at each step a corner in a clockwise order

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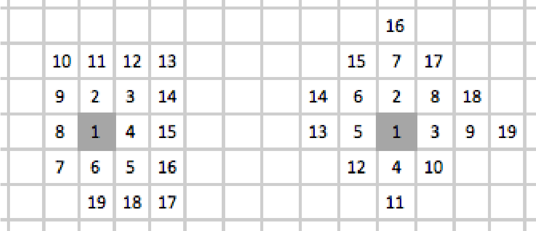

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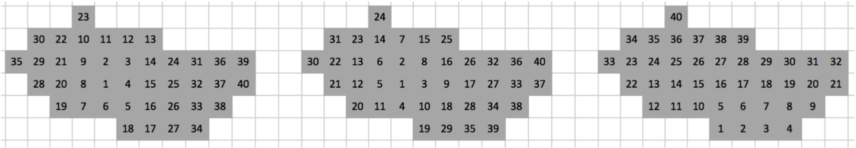
This construction leads to an octogonal shape digitally convex polyomino.

Strate construction



1. It consists to take the lowest unit squares from the left to the right,

2. it continues to the second row in a correct order.

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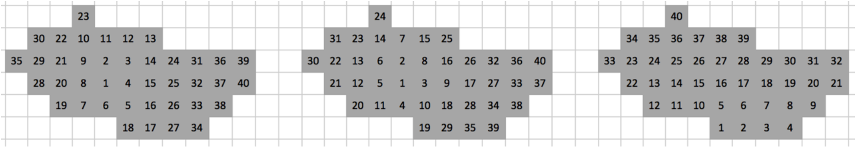

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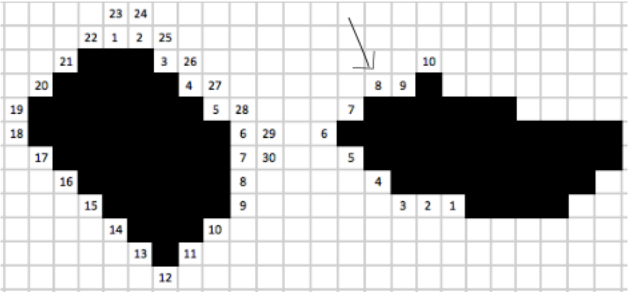
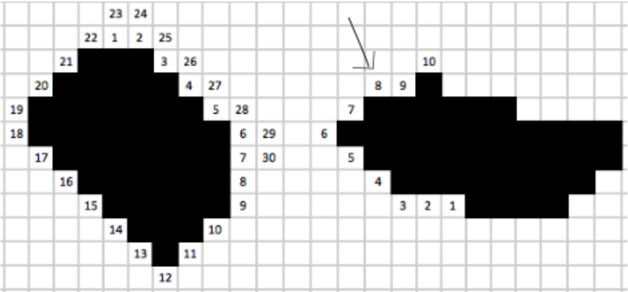
BUT!!



- the spiral method works only when we take special octogones,
- the convexity property disappears at some step, in the general case.



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Question:How can we solve this problem?

17e journées montoises d'informatique théorique

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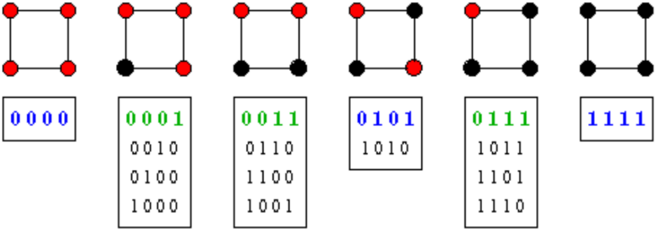
Outline

4. Combinatorics on words
5. Inflation of a DC polyomino with discrete geometry
6. Conclusion

Lyndon words

Roger Lyndon in 1954, introduced the *Standard Lexicographic sequences*.



Definition

A $w \in A^+$ is a Lyndon word if it is the smallest between all its conjugates with respect to the lexicographic order.

Lyndon factorization

Theorem (Chen-Fox 1954)

Every non empty word w admits a unique factorization as a lexicographically decreasing sequence of Lyndon words.
$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$, *s.t* $l_1 >_l l_2 >_l \cdots l_k$ *where* $n_i \geq 1$ *and* l_i *are Lyndon words.*

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Example

Let $w = 100101100101010$, the Lyndon factorization is given as follows:

$$w = (1)(001011)(0010101)(0).$$

Christoffel words

1. The closest path to the line segment.
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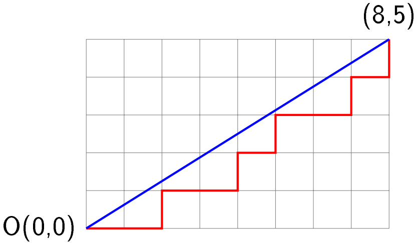

Figure: The line segment from $O(0,0)$ to $(8,5)$ has the following Christoffel word: $w = 0010010100101$ of slope $\frac{5}{8}$.

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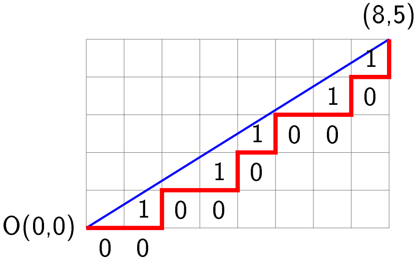
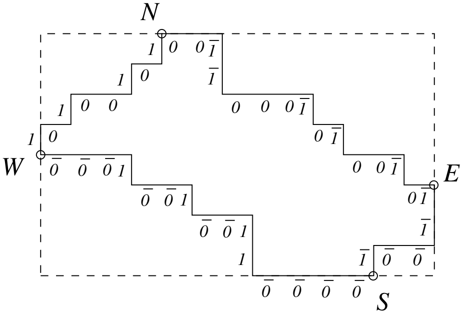


Figure: The line segment from $O(0,0)$ to $(8,5)$ has the following Christoffel word:$w = 0010010100101$ of slope $\frac{5}{8}$.

- The border of a DC polyomino S, Bd(S), is the 4-connected path that follows clockwise the points of S that are 8-adjacent to some points not in S.
- This path is a word in $\{0, 1, \overline{0}, \overline{1}\}^*$, starting by convention from the leftmost lower point considered in the clockwise order.



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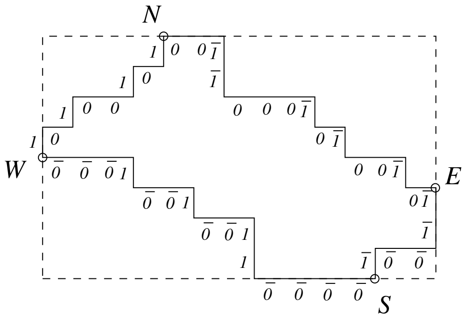

The word $w \in A$ coding the WN path is $w = 10100101$.

Lyndon + Christoffel = Digitally Convex ⋆

S. Brlek [a], J.-O. Lachaud [b], X. Provençal [a], C. Reutenauer [a],

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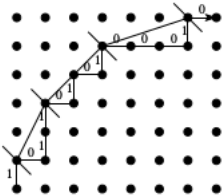
Theorem (B,L,P,R 2010)

A word w is WN -convex iff its unique Lyndon factorization $l_1^{n_1} l_2^{n_2} \ldots l_k^{n_k}$ is such that all l_i are Christoffel words.

Example

Consider the following WN-convex path $v = 1011010100010$.



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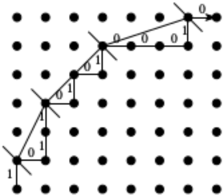

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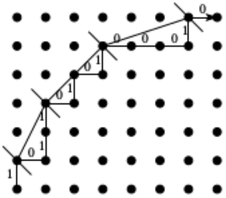

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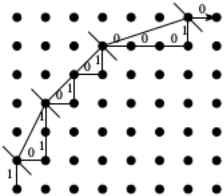

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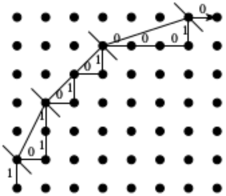

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The Christoffel words are arranged in a decreasing order of slopes.

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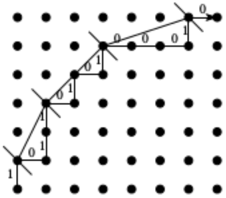

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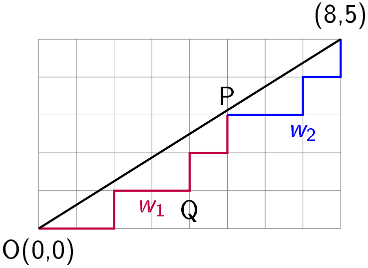
$$\frac{1}{0} > \frac{2}{1} > \frac{1}{1} > \frac{1}{3} > \frac{0}{1}$$

Outline

5. Inflation of a DC polyomino with discrete geometry

Particular points



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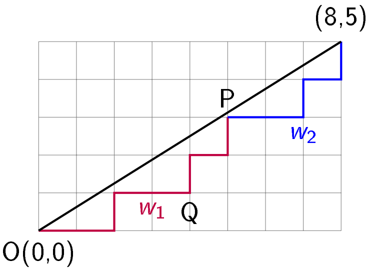

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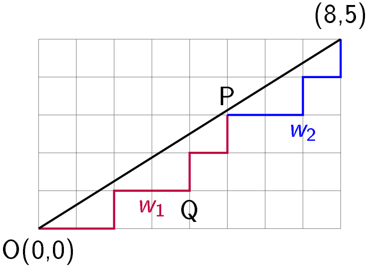


By Borel and Laubie (BL1993), we have the uniqueness of the following two points:

1. P the closest point to the line segment on the Christoffel path. (Standard factorization)
2. Q the furthest point from the line segment on the Christoffel path. (Palindromic factorization)

Split operator

Let w be a Christoffel word of length l.

The furthest point of the path from the line segment is at position k.

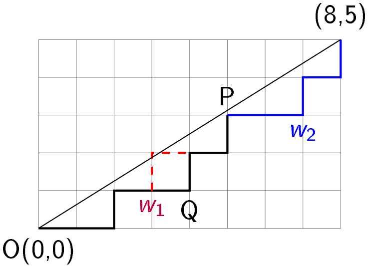
At this position, we have: $w[k] = 0$ and $w[k+1] = 1$.

The split operator exchanges the factor 01 into 10

Proposition (Tarsissi et al. 17)

The words $w^+ = w[1, k-1]1$ and $w^- = 0w[k+1, l]$, are two Christoffel words. We have: $w^+ > w^-$.

Example



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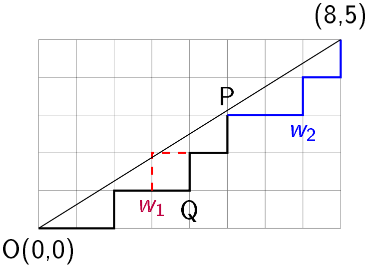

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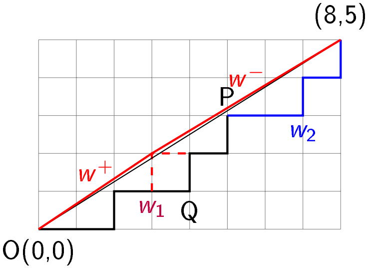

1. The Christoffel word of slope $\frac{5}{8}$ is given by: $w = (w_1, w_2)$,
2. $w^+ = 00101$ and $w^- = 00100101$ are Christoffel words with:

$$\frac{2}{3} > \frac{3}{5}.$$

Inflation of a DC by adding one unit square

Let $u = \ldots \ell" \, \ell \, \ell' \ldots$ be a part of the *WN* path of DC.

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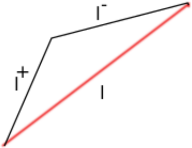

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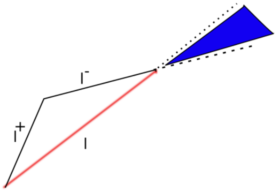

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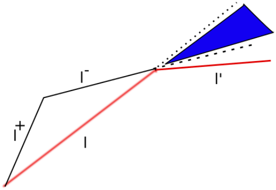

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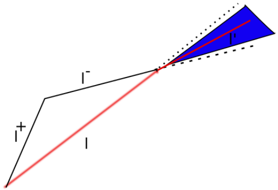

2. we must choose the longest word to split

3. The convexity is not conserved by adding a single point.

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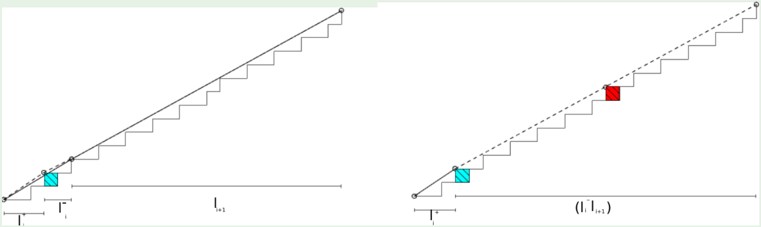
Example

Let $slope(l_i) = \frac{3}{5} \geq slope(l_{i+1}) = \frac{11}{20}$.
The spliting of (l_i) gives $\frac{2}{3} > \frac{1}{2} < \frac{11}{20}$.



3. The convexity is not conserved by adding a single point.

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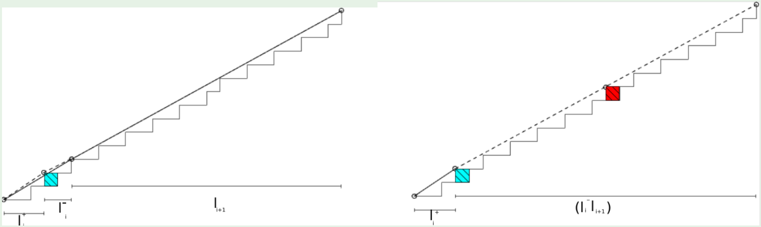

$$
\begin{aligned}
l_i^- . l_{i+1} \quad &= 001.0010010010010\textcolor{red}{01}0100100100100101 \\
&= 0010010010010010\textcolor{red}{10}0100100100100101.
\end{aligned}
$$

Conclusion

Corollary

The cases 1 or 2 occur essentially when:

a) $\ell' \leq \ell^{-}$ or $\ell' = \ell^{-k}\ell$ for some positive integer k and a Christoffel word ℓ,

b) $\ell" \geq \ell^{+}$ or $\ell" = \ell\ell^{+k'}$ for some positive integer k' and a Christoffel word ℓ.

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By choosing for ℓ the longest word of WN-path, then automatically we have:

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There always exists in each of the four paths describing C_1 at least one Christoffel word corresponding to the cases 1 or 2.

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The inlation by keeping $C_1 \subset C_2$ is always possible.

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Thank You