Inflation of digitally convex polyominoes

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Outline

1. Transformations of digitally convex polyominoes
2. Intuitive approach
3. Another approach with combinatorics on words
The continuous version of convexity

**Definition**

A set in Euclidean geometry is convex if and only if for any pair of points $p_1$, $p_2$ in a region $R$, the line segment joining them is completely included in $R$. 
What about the **discrete** version?
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Inflate and deflate of convexes

It can be done in the slowlest way by passing from:

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\[ MA = \lambda \ AB; \lambda \in [0, 1] \]

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1. An empty set to a convex
2. A small $C_1$ into a bigger $C_2$

How can we do it in the digital case?

As in the continuous case, we have the expansion from an interior pixel by adding pixel by pixel.
Deflate of a DC polyomino $P$

The **corners** of $P$ are the pixels such that a vertex is an angle of the convex hull.

To deflate $P$, we do it step by step by removing one unit square at each time.
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However, it does not give a practical way to choose the unit square that we must add at each step.
Outline

3. Inflate DC polyominoes
### The spiral construction

1. By adding a corner around the polyomino from a unit square polyomino

2. by adding at each step a corner in a clockwise order

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The spiral construction

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This construction leads to an octogonial shape digitally convex polyomino.
It consists to take the lowest unit squares from the left to the right,

it continues to the second row in a correct order.
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**BUT!!**
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the convexity property disappears at some step, in the general case.

**Question**: How can we solve this problem?
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10-14 sept. 2018 Talence (France)
Outline

4. Combinatorics on words
5. Inflation of a DC polyomino with discrete geometry
6. Conclusion
Lyndon words

Roger Lyndon in 1954, introduced the *Standard Lexicographic sequences*.

**Definition**

A \( w \in A^+ \) is a Lyndon word if it is the smallest between all its conjugates with respect to the lexicographic order.
Lyndon factorization

**Theorem (Chen-Fox 1954)**

*Every non empty word $w$ admits a unique factorization as a lexicographically decreasing sequence of Lyndon words.*

$$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}, \text{ s.t } l_1 >_l l_2 >_l \cdots >_l l_k \text{ where } n_i \geq 1 \text{ and } l_i \text{ are Lyndon words.}$$
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Example

Let $w = 100101100101010$, the Lyndon factorization is given as follows:

$w = (1)(001011)(0010101)(0)$. 
Christoffel words

1. The closest path to the line segment.

2. There are no points of $\mathbb{Z} \times \mathbb{Z}$ between the path and line segment.
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Figure: The line segment from $O(0,0)$ to $(8,5)$ has the following Christoffel word: $w = 0010010100101$ of slope $\frac{5}{8}$.
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Figure: The line segment from $O(0,0)$ to $(8,5)$ has the following Christoffel word: $w = 00100100101$ of slope $\frac{5}{8}$.
The border of a DC polyomino $S$, $Bd(S)$, is the 4-connected path that follows clockwise the points of $S$ that are 8-adjacent to some points not in $S$.

This path is a word in $\{0, 1, \overline{0}, \overline{1}\}^*$, starting by convention from the leftmost lower point considered in the clockwise order.
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The word \( w \in A \) coding the WN path is \( w = 10100101 \).
Lyndon + Christoffel = Digitally Convex

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Theorem (B,L,P,R 2010)

A word $w$ is WN -convex iff its unique Lyndon factorization $l_1^{n_1} l_2^{n_2} \ldots l_k^{n_k}$ is such that all $l_i$ are Christoffel words.
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![Diagram of WN-convex path]

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\[
\begin{align*}
1 & > 2 \\
\bar{0} & > \bar{1} > \bar{1} > \bar{1} \\
\bar{3} & > \bar{1}
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5 Inflation of a DC polyomino with discrete geometry
Particular points

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By Borel and Laubie (BL1993), we have the uniqueness of the following two points:

1. P the closest point to the line segment on the Christoffel path. (Standard factorization)
2. Q the furthest point from the line segment on the Christoffel path. (Palindromic factorization)
Split operator

Let $w$ be a Christoffel word of length $l$.

The furthest point of the path from the line segment is at position $k$.

At this position, we have: $w[k] = 0$ and $w[k + 1] = 1$.

The split operator exchanges the factor 01 into 10

**Proposition (Tarsissi et al. 17)**

The words $w^+ = w[1, k - 1]1$ and $w^- = 0w[k + 1, l]$, are two Christoffel words. We have: $w^+ > w^-$.
The Christoel word of slope \((8,5)\) is given by:

\[ w = (w_1, w_2), \]

where \(w_1 + w_2 = 00101\) and \(w_2 = 00100101\) are Christoel words with:

\[ w_1 \neq 3, w_2 \neq 5. \]
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2 $w^+ = 00101$ and $w^- = 00100101$ are Christoffel words with:

$$\frac{2}{3} > \frac{3}{5}.$$
Inflation of a DC by adding one unit square

Let $u = \ldots \ell'' \ell \ell' \ldots$ be a part of the $WN$ path of DC.

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2. we must choose the longest word to split
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Example

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\[ l_i \cdot l_{i+1} = 001.0010010010010010010101010110100100100100100100101. \]
Conclusion

Corollary

*The cases 1 or 2 occur essentially when:*

a) $\ell' \leq \ell^{-}$ or $\ell' = \ell^{-k}\ell$ for some positive integer $k$ and a Christoffel word $\ell$,

b) $\ell'' \geq \ell^{+}$ or $\ell'' = \ell\ell^{+k'}$ for some positive integer $k'$ and a Christoffel word $\ell$. 
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By choosing for \( \ell \) the longest word of WN-path, then automatically we have:

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There always exists in each of the four paths describing \( C_1 \) at least one Christoffel word corresponding to the cases 1 or 2.
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The inflation by keeping \( C_1 \subset C_2 \) is always possible.
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Thank You