Fixed points of Sturmian morphisms and their derivated words

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if we take any prefix, we obtain again the Fibonacci word

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if **u** is a fixed point of a primitive morphism, then the set of all its derivated words is finite

and yet again by **[F. Durand, 1998]**
all such derivated words are fixed by a primitive morphism

Known results and our questions

[I. M. Araújo, V. Bruyère, 2005] : slopes of derivated words of standard/characteristic/homogeneous Sturmian words

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Our question:
what are the morphisms fixing the derivated words and what is their number?

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$$\mathrm{Der}(\mathbf{u}) = \{ \text{ derivated word of } \mathbf{u} \text{ with respect to } w \colon\ w \text{ is a prefix of } \mathbf{u} \}$$

Sturmian monoid

We work with these four elementary Sturmian morphisms:

$$\varphi_a : \begin{cases} 0 \to 0 \\ 1 \to 10 \end{cases} \quad \varphi_b : \begin{cases} 0 \to 0 \\ 1 \to 01 \end{cases} \quad \varphi_\alpha : \begin{cases} 0 \to 01 \\ 1 \to 1 \end{cases} \quad \varphi_\beta : \begin{cases} 0 \to 10 \\ 1 \to 1 \end{cases}$$

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Set $\mathcal{M} = \langle \varphi_a, \varphi_b, \varphi_\alpha, \varphi_\beta \rangle$.

Notation: $\varphi_w = \varphi_{w_0} \varphi_{w_1} \cdots \varphi_{w_{|w|-1}}$, e.g., $\varphi_{a\alpha b} = \varphi_a \varphi_\alpha \varphi_b$

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The monoid $\mathcal{M}$ has presentation

$$\varphi_{\alpha a^k \beta} = \varphi_{\beta b^k \alpha} \quad \text{and} \quad \varphi_{a\alpha^k b} = \varphi_{b\beta^k a}.$$

[P. Séébold, 1991] [C. Kassel, C. Reutenauer, 2007]

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The mapping Δ

Let $w \in \{a, b, \alpha, \beta\}^* \setminus \{a, \alpha\}^*$ be the normalized name of a morphism ψ, i.e., $\psi = \varphi_w$. We set

$$\Delta(w) = \begin{cases} N(w' a^k \beta) & \text{if } \ w = a^k \beta w', \\ N(w' \alpha^k b) & \text{if } \ w = \alpha^k b w'. \end{cases}$$

Example

Consider the morphism $\psi = \varphi_w$, where $w = N(w) = \beta\alpha aa\alpha$, and apply repeatedly the transformation Δ on w.

$$
\begin{aligned}
w &= \beta\alpha aa\alpha \\
\Delta(w) &= \alpha aa\alpha\beta \\
\Delta^2(w) &= bb\alpha\alpha\beta \\
\Delta^3(w) &= b\beta\alpha\alpha b \\
\Delta^4(w) &= \beta\alpha\alpha bb \\
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The 5 fixed points of the morphisms
$\varphi_{\Delta(w)}, \varphi_{\Delta^2(w)}, \varphi_{\Delta^3(w)}, \varphi_{\Delta^4(w)}, \varphi_{\Delta^5(w)}$ are exactly the five derivated words of the fixed point of ψ.

Morphisms with unique fixed points

Theorem

Let $\psi \in \langle \varphi_a, \varphi_b, \varphi_\alpha, \varphi_\beta \rangle \setminus \langle \varphi_a, \varphi_\alpha \rangle$ be a primitive morphism w be its normalized name.

Denote **u** *the fixed point of ψ.*

The word **x** *is (up to a permutation of letters) a derivated word of* **u** *with respect to one of its prefixes if and only if* **x** *is the fixed point of the morphism $\varphi_{\Delta^j(w)}$ for some $j \geq 1$.*

Morphisms with two fixed points

Given a finite word u, we define the **cyclic shift** of $u = u_0 u_1 \cdots u_{n-1}$ to be the word

$$\mathrm{cyc}(u) = u_1 u_2 \cdots u_{n-1} u_0.$$

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Theorem

Let $\psi = \varphi_w$ be a primitive morphism with $w \in \{a, \alpha\}^*$ and a be the first letter of w.

(0) *Let $\mathbf{u}$ be the fixed point of ψ starting with 0. Denote $v = b^{-1}N(wb) \in \{a, \beta\}^*$. We have $\mathrm{Der}(\mathbf{u}) = \{\mathbf{v}\} \cup \mathrm{Der}(\mathbf{v})$, where $\mathbf{v}$ is the unique fixed point of the morphism φ_v.*

(1) *Let $\mathbf{u}$ be the fixed point of ψ starting with 1. Put $v = \mathrm{cyc}(w)$. We have $\mathrm{Der}(\mathbf{u}) = \mathrm{Der}(\mathbf{v})$, where $\mathbf{v}$ is the fixed point of the morphism φ_v starting with 1.*

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If b is the first letter: $a \leftrightarrow b$, $\alpha \leftrightarrow \beta$ and $0 \leftrightarrow 1$.

Moreover, in the case $w \in \{a, \alpha\}^*$ we know that the Sturmian words in $\mathrm{Der}(\mathbf{u})$ (fixed by an element of $\langle a, \beta \rangle$ or $\langle \alpha, b \rangle$) have intercept 0 **[M. Dekking, 2017]**

Bounds on the number of derivated words

If $w \in \{a, b, \alpha, \beta\}^* \setminus \{a, \alpha\}^*$ is the normalized name of a primitive morphism $\psi = \varphi_w$ and $\mathbf{u}$ is a fixed point of ψ, then

$$1 \leq \#\mathrm{Der}(\mathbf{u}) \leq 3|w| - 4\,. \tag{2}$$

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For $n \geq 2$ set $w' = \beta^{n-2}a\alpha$ and $w'' = \alpha^{n-1}\beta$. We have

(i) $\varphi_{w'}$ and $\varphi_{w''}$ are not powers of other Sturmian morphisms,

(ii) for the fixed points $\mathbf{u}'$ and $\mathbf{u}''$ of the morphisms $\varphi_{w'}$ and $\varphi_{w''}$, the lower resp. the upper bound in (2) is attained.

Exact counts - mappings F and E

$F : \{a, b, \alpha, \beta\}^* \mapsto \{a, b, \alpha, \beta\}^*$ determined by

$$F(a) = \alpha, \quad F(\alpha) = a, \quad F(b) = \beta, \quad F(\beta) = b,$$

and we set

$$\mathrm{cyc}_{\mathrm{F}}(w_1 w_2 w_3 \cdots w_n) = w_2 w_3 \cdots w_n F(w_1)$$

for a finite word $w_1 w_2 w_3 \cdots w_n$.

E is the morphism over $\{0, 1\}^*$ determined by

$$E(0) = 1 \quad \text{and} \quad E(1) = 0.$$

Exact counts - standard Sturmian morphisms

Let **u** be a fixed point of a standard Sturmian morphism ψ which is not a power of any other Sturmian morphism ($\psi \in \langle b, \beta \rangle \cup \langle b, \beta \rangle \circ E$).

(i) If $\psi = \varphi_w$ (i.e., $\psi \in \langle \varphi_b, \varphi_\beta \rangle$), then **u** has $|w|$ distinct derivated words, each of them (up to a permutation of letters) is fixed by one of the morphisms

$$\varphi_{v_0}, \varphi_{v_1}, \varphi_{v_2}, \ldots, \varphi_{v_{|w|-1}},$$

where $v_k = \mathrm{cyc}^k(w)$ for $k = 0, 1, \ldots, |w| - 1$.

(ii) If $\psi = \varphi_w \circ E$ (i.e., $\psi \in \langle \varphi_b, \varphi_\beta \rangle \circ E$), then **u** has $|w|$ distinct derivated words, each of them (up to a permutation of letters) is fixed by one of the morphisms

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where $v_k = \mathrm{cyc}_F^k(w)$ for $k = 0, 1, \ldots, |w| - 1$.

Can be inferred from **[I. M. Araújo, V. Bruyère, 2005]** .

Exact counts - $\Psi \in \langle \varphi_a, \varphi_\alpha \rangle$

Let $w \in \{\alpha, a\}^*$ be the normalized name of a primitive morphism ψ such that the letter a is a prefix of w. Moreover, assume that ψ is not a power of any other Sturmian morphism.

(i) The fixed point of ψ starting with 0 has exactly $1 + |w|_\alpha$ distinct derivated words.

(ii) The fixed point of ψ starting with 1 has exactly $1 + |w|_a$ distinct derivated words.

If b is the first letter: $a \leftrightarrow b$, $\alpha \leftrightarrow \beta$ and $0 \leftrightarrow 1$.

Open questions

We looked at derivated words only with respect to a prefix. For standard/characteristic Sturmian words it is easy to extend the results to any factor.

Q1: Non-standard words?

Exact counts for other Sturmian morphisms:
Q2: a description when a normalized name is corresponds to a power of a morphism is needed.

Q3: Generalizations - see **[V. Berthé, F. Dolce, F. Durand, J. Leroy, D. Perrin, *Rigidity and substitutive dendric words*]**
Extension to other S-adic sequences requires presentation of the morphism monoid.

Thank you