Simple algorithms and fast-growing complexity for well-structured systems

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Based on joint work with Sylvain Schmitz, Pierre Chambart, Prateek Karandikar, Simon Halfon, K. Narayan Kumar, Alain Finkel, ..

Background & motivations

- Well-Structured Systems (WSTS) are a family of infinite-state models where safety, inevitability, etc., properties are decidable
- There, decidability relies on the fact that states are well-quasi-ordered and uses generic algorithms.
- WSTS invented by Finkel (1987), developed and popularized by Abdulla & Jonsson, Finkel & Schnoebelen, etc. (1996–2005).
- First used with counters, queues, gap-order constraints, etc.
- The family encompasses many kinds of models: distributed systems, counter systems, out-of-order memory, communication protocols, automata for logic, . . . (still growing).
- In 2017, WSTS recognized as a fundamental contribution by the Computer-Aided Verification community.

Outline of the Talk

- Part 0: **Basics of WQOs.**
- Part 1: **Basics of WSTS.**
- Part 2: **Verifying WSTS.**
- Part 3: **Assessing Complexity.**

Part 0

Basics of WQOs

QUASI-ORDERINGS

We consider orderings, like e.g.

$$(\mathbb{Z}, \leqslant) \qquad \begin{array}{c} \vdots \\ \vee \\ 2 \\ \vee \\ 1 \\ \vee \\ 0 \\ \vee \\ -1 \\ \vee \\ \vdots \end{array}$$

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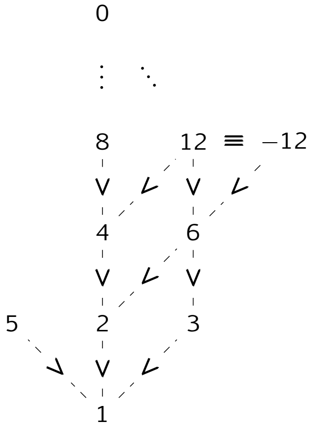
$(\mathbb{Z}, \leqslant_{\mathsf{div}})$



Quasi-orderings are more robust. If $(X, \leqslant)$ is an ordering, $(\mathcal{P}(X), \sqsubseteq)$ is in general not antisymmetric

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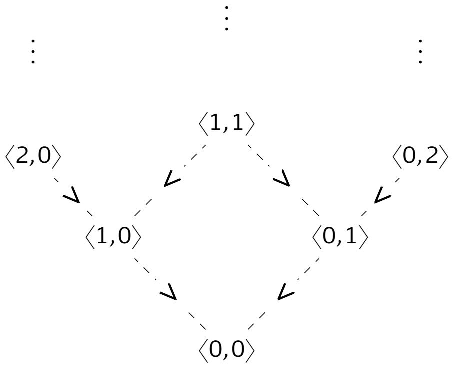
$$(\mathbb{N}^2, \leqslant_\times)$$



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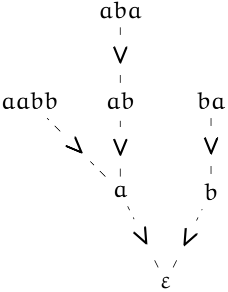

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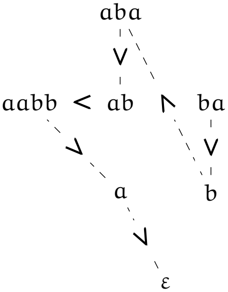

$(A^*, \leqslant_{\text{lex}})$ is a total/linear ordering that contains/extends $(A^*, \leqslant_{\text{pref}})$

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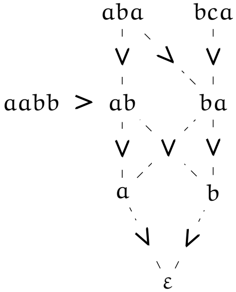

$\leqslant_*$ is the subsequence/subword ordering. It extends $\leqslant_{\text{pref}}$

WELL-QUASI ORDERINGS (WQO)

A WQO is a quasi-ordering $(X, \leqslant)$

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$(\mathbb{N}^2, \leqslant_\times)$

$(A^*, \leqslant_*)$

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$(A^*, \leqslant_{\text{lex}})$

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A WQO is a quasi-ordering $(X, \leqslant)$ that is well-founded

– WF: no infinite decreasing sequence $x_0 > x_1 > x_2 > \cdots$

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WELL-QUASI ORDERINGS (WQO)

A WQO is a quasi-ordering $(X, \leqslant)$ that is well-founded and has the finite antichain property

– WF: no infinite decreasing sequence $x_0 > x_1 > x_2 > \cdots$

– FAC: no infinite set $\{x_0, x_1, x_2, \ldots\}$ of pairwise incomparable elements

$(\mathbb{Z}, \leqslant)$ ✗ $\quad (\mathbb{N}^2, \leqslant_\times)$

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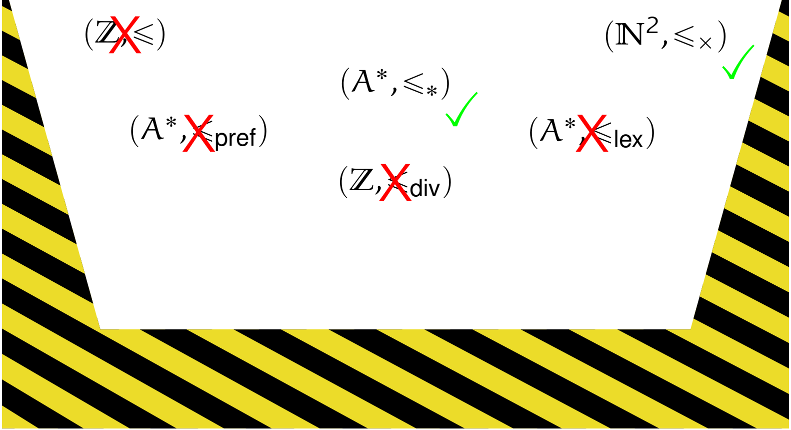


Other WQO facts

Many characterizations: $(X, \leqslant)$ is wqo iff

- it is WF and FAC;
- every infinite sequence $x_0, x_1, x_2, \ldots$ is good, i.e. contains an increasing pair $x_i \leqslant x_j$ (for some $i < j$);
- every infinite sequence $x_0, x_1, x_2, \ldots$ is perfect, i.e. contains an infinite increasing subsequence $x_{i_0} \leqslant x_{i_1} \leqslant x_{i_2} \leqslant \cdots$ (with $i_0 < i_1 < i_2 < \cdots$);
- every linearization $(X, \leqslant')$ of $\leqslant$ is a wellorder.

Many ways to construct WQOs:

- Cartesian products, powersets, ..
- sequences, trees, graphs.

Part 1

Basics of WSTS

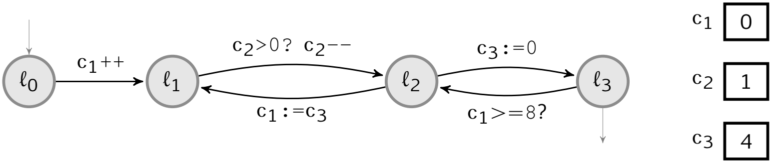
Well-structured systems

In program verification, wqo's appear prominently in well-structured systems (WSTS).

Def. A WSTS is a system $(S, \rightarrow, \leqslant)$ where

1. $(S, \rightarrow)$ with $\rightarrow \subseteq S \times S$ is a transition system
2. the set of states $(S, \leqslant)$ is wqo, and
3. the transition relation is compatible with the ordering (also called "monotonic"): $s \rightarrow t$ and $s \leqslant s'$ imply $s' \rightarrow t'$ for some $t' \geqslant t$

SOME WSTS'S: MONOTONIC COUNTER MACHINES



A run of M: $(\ell_0,0,1,4) \to (\ell_1,1,1,4) \to (\ell_2,1,0,4) \to (\ell_3,1,0,0)$

Ordering states: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$.
This is wqo as a product of wqo's: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$

Compatibility: easily checked when guards are upward-closed and assignments are monotonic functions of the variables.

Related models: Petri nets; vector additions systems; broadcast protocols; etc.

NB. Minsky (counter) machines have zero-tests, hence don't enjoy monotonicity

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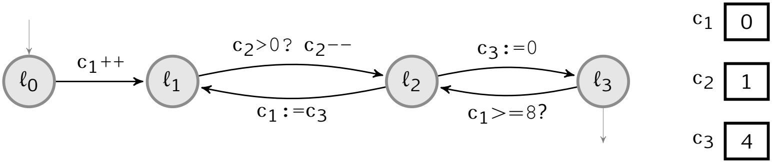

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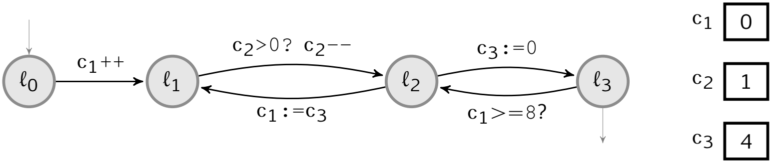

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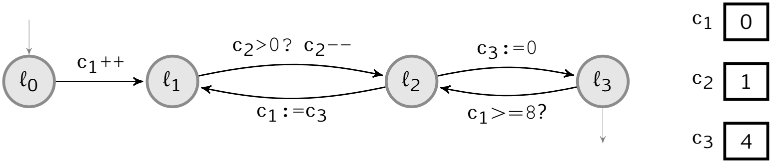

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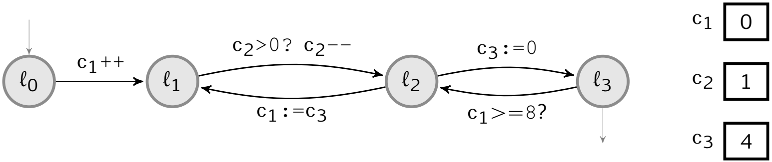

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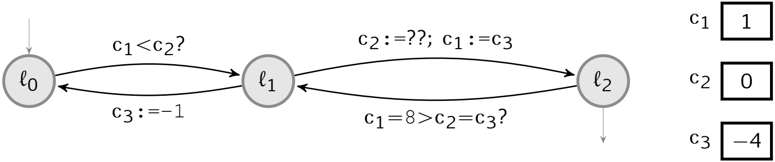

Guards: comparisons between counters and constants

Updates: assignments with counter values, constants, & "??"

One does not use $\leqslant_\times$ to compare states!! Rather

$$(a_1,\ldots,a_k)\leqslant_{\text{sparse}}(b_1,\ldots,b_k)$$

$$\overset{\text{def}}{\Leftrightarrow} \forall i,j=1,\ldots,k: \big(a_i \leqslant a_j \text{ iff } b_i \leqslant b_j\big) \wedge \big(|a_i-a_j| \leqslant |b_i-b_j|\big)\,.$$

Fact. $(\mathbb{Z}^k,\leqslant_{\text{sparse}})$ is wqo

Compatibility: We use

$$(\ell,a_1,\ldots,a_k)\leqslant(\ell',b_1,\ldots,b_k) \overset{\text{def}}{\Leftrightarrow} \tag{1}$$

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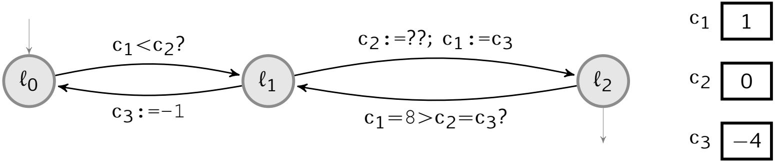

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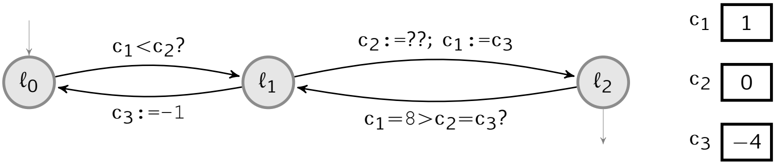

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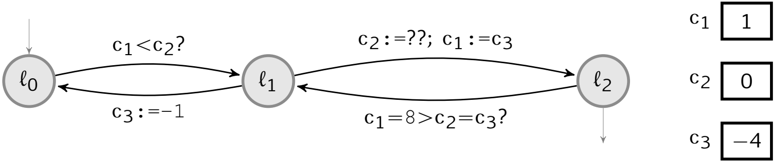

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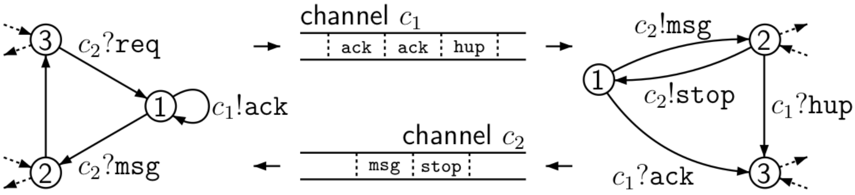
Some WSTS's: Lossy Channel Systems (LCS)



A state $s = (\ell_1, \ell_2, w_1, w_2)$ with $w_i \in A^*$.
E.g., $w_1 = \texttt{hup.ack.ack}$.

Reliable steps: $s \to_{\text{rel}} s'$ read in front of channels, write at end (FIFO)

Lossy steps: messages may be lost nondeterministically

$$s \to s' \overset{\text{def}}{\Leftrightarrow} s \geqslant_* t \to_{\text{rel}} t' \geqslant_* s' \text{ for some } t, t' \in S$$

where $(S, \sqsubseteq)$ is the wqo $(Loc_1, =) \times (Loc_2, =) \times (A^*_{c_1}, \leqslant_*) \times (A^*_{c_2}, \leqslant_*)$

A model useful for concurrent protocols but also timed automata, metric temporal logic, products of modal logics, ...

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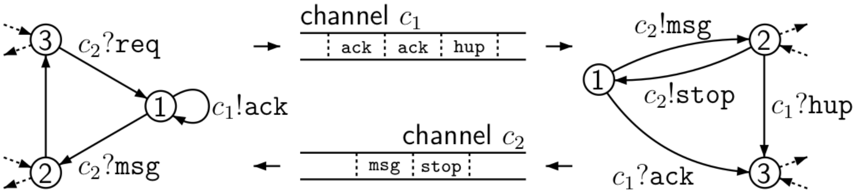

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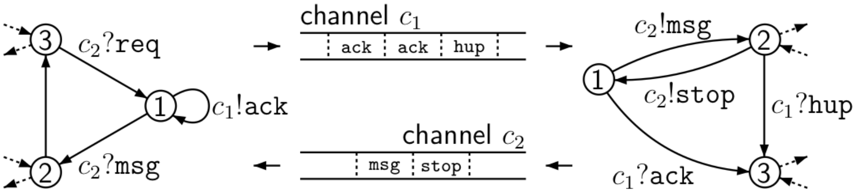

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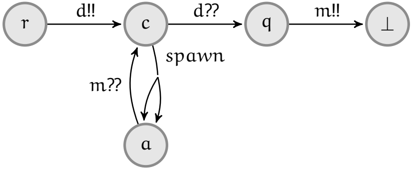
SOME WSTS'S: BROADCAST PROTOCOLS

Broadcast protocols (Esparza, Finkel, Mayr 1999), aka population protocols, are dynamic & distributed collections of finite-state processes communicating via brodcasts (and rendez-vous, not shown here).



A configuration collects the local states of all processes. E.g., $s = \{c, r, c\}$, also denoted $\{c^2, r\}$.

Steps:

$\{c^2, q, r\} \xrightarrow{s(pawn)} \{a^2, c, q, r\} \xrightarrow{s} \{a^4, q, r\} \xrightarrow{m} \{c^4, r, \perp\} \xrightarrow{d} \{c, q^4, \perp\}$

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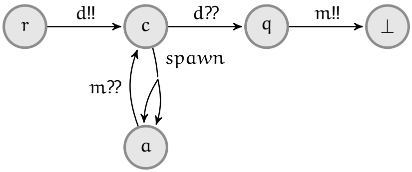

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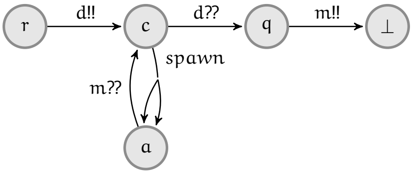
PROVING TERMINATION

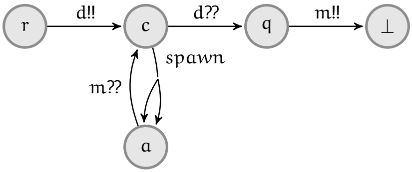


- This protocol has no infinite runs

Proof. Write $s = \{r^{n_1}, q^{n_2}, c^{n_3}, a^*, \perp^*\}$.
In any step $s \to s'$ the triple $\langle n_1, n_2, n_3 \rangle$ decreases in the lexicographic ordering

This is the pattern for proofs of termination: one invents a well-founded measure that decreases with every step

Proving Termination



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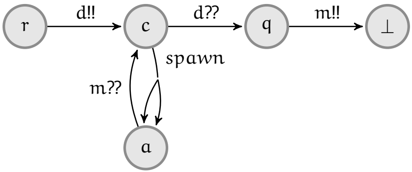

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BRODCAST PROTOCOLS ARE WSTS

1. Order the configurations by multiset inclusion, e.g.,
$\{c,q\} \subseteq \{c^2,r,q\}$

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Proof. Case analysis: is $s \to t$ an internal move? or a spawning step? or a broadcasting? or a rendez-vous?

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Part 2

Verification of WSTS

Deciding Termination for WSTS

Lem. [Finite Witnesses for Infinite Runs]
A WSTS $\mathcal{S}$ has an infinite run from s_{init} **iff** it has a finite run from s_{init} that is a good sequence.

Recall: $s_0, s_1, s_2, \ldots, s_n$ is good $\overset{\text{def}}{\Leftrightarrow}$ there exist $i < j$ s.t. $s_i \leqslant s_j$

Corollary. One may decide Termination by enumerating all finite runs from s_{init} until a good sequence is encountered.
If all runs are bad, the enumeration will eventually exhaust them

NB: This requires some minimal effectiveness assumptions on the WSTS, e.g., that the ordering is decidable

Algorithm extends and allows deciding inevitability, finiteness, and regular simulation
A similar algorithm allows deciding safety properties

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Proof. $\Rightarrow$: by definition since $\leqslant$ is wqo
$\Leftarrow$: good finite run $s_0 \xrightarrow{*} s_i \xrightarrow{+} s_j$ can be extended by simulating $s_i \xrightarrow{+} s_j$ from above: $s_j \xrightarrow{+} s_{2j-i}$, then $s_{2j-i} \xrightarrow{+} s_{3j-2i}$, etc.

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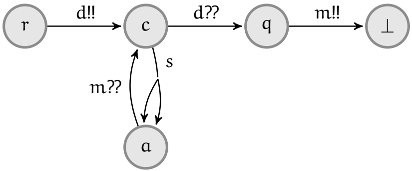
Part 3a

Complexity: Upper Bounds

Broadcast Protocols Take Their Time



"Doubling" run: $\{c^{n}, q, (\bot^*)\} \xrightarrow{s^n} \{a^{2n}, q, (\bot^*)\} \xrightarrow{m} \{c^{2n}, (\bot^*)\}$

Building up: $\{c^{2^0}, q^{n}, r\} \xrightarrow{s^{2^0} m} \{c^{2^1}, q^{n-1}, r\} \xrightarrow{s^{2^1} m} \{c^{2^2}, q^{n-2}, r\} \rightarrow$
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Then: $\{c, q, r^{n}\} \xrightarrow{*} \{c, q^{2^n}, r^{n-1}\} \xrightarrow{*} \{c, q^{\text{tower}(n)}\}$

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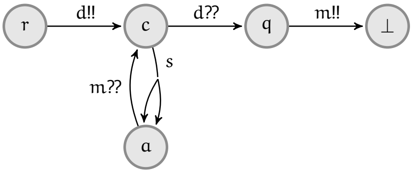

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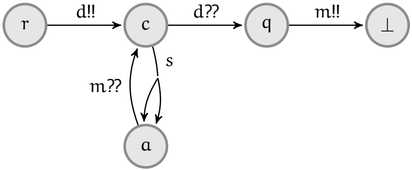

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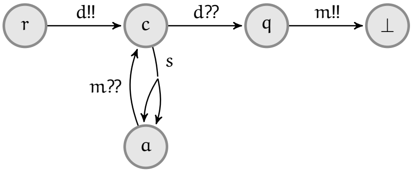

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⇒ Runs of terminating systems may have nonelementary lengths

⇒ Running time of generic algorithm verifying termination is not elementary for broadcast protocols

THE FAST-GROWING HIERARCHY

An ordinal-indexed family $(F_\alpha)_{\alpha \in Ord}$ of functions $\mathbb{N} \to \mathbb{N}$

$$F_0(x) \stackrel{\text{def}}{=} x + 1 \qquad F_{\alpha+1}(x) \stackrel{\text{def}}{=} \overbrace{F_\alpha(F_\alpha(\dots F_\alpha}^{x+1}(x)\dots))$$

$$F_\omega(x) \stackrel{\text{def}}{=} F_{x+1}(x)$$

gives $F_1(x) \sim 2x$, $F_2(x) \sim 2^x$, $F_3(x) \sim \text{tower}(x)$ and $F_\omega(x) \sim \text{ACKERMANN}(x)$, the first F_α that is not primitive recursive.

$F_\lambda(x) \stackrel{\text{def}}{=} F_{\lambda_x}(x)$ for λ a limit ordinal with a fundamental sequence $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda$.

E.g. $F_{\omega^2}(x) = F_{\omega\cdot(x+1)}(x) = F_{\omega\cdot x+x+1}(x) = \overbrace{F_{\omega\cdot x+x}(F_{\omega\cdot x+x}(..F_{\omega\cdot x+x}}^{x+1}(x)..))$

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When analyzing the termination algorithm, the main question is "how long can a bad sequence be?"

WQO-theory only says that a bad sequence is finite

One can exhibit arbitrarily long bad sequences. E.g. over $(\mathbb{N}^k, \leqslant_\times)$:

— 999, 998, . . . , 1, 0

— (2,2), (2,1), (2,0), (1,999), . . . , (1,0), (0,999999999), . . .

Two tricks: unbounded start element, or unbounded increase in a step

The runs of broadcast protocols don't have unbounded increases, and the starting configuration is the input of our problem

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CONTROLLED BAD SEQUENCES

Def. A sequence $x_0, x_1, \ldots$ is controlled $\overset{\text{def}}{\Leftrightarrow}$ $|x_i| \leqslant g^i(n_0)$ for all $i = 0, 1, \ldots$

Here the control is the pair (n_0, g) of $n_0 \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$.

Fact. For a fixed wqo $(A, \leqslant, |.|)$ and control (n_0, g), there is max length on controlled bad sequences (Kőnig's Lemma again)

Write $L_{g,A}(n_0)$ for this maximum length.

Length Function Theorem for $(\mathbb{N}^k, \leqslant_\times)$ [McAloon,Schmitz & S.,...]

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— g'_k and $L_{g,\mathbb{N}^k}$ are in $\mathscr{F}_{k+m-1}$ for g in $\mathscr{F}_m$

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APPLYING TO BROADCAST PROTOCOLS

The runs explored by the Termination algorithm are controlled with $|s_{\text{init}}|$ and $Succ : \mathbb{N} \to \mathbb{N}$.

$\Rightarrow$ Time/space bound in $\mathscr{F}_{k-1}$ for broadcast protocols with k states, and in $\mathscr{F}_{\omega}$ when k is not fixed.

NB. Similar controls for the backward-chaining Coverability algorithm: $|s_{target}|$ and $Succ$.

$\Rightarrow \cdots$ *same upper bounds* $\cdots$

This is a general situation:
— WSTS model (or WQO-based algorithm) provides A and g
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More Length Function Theorems

For finite words with embedding, $L_{\Sigma*}$ is in $\mathscr{F}_{\omega^{|\Sigma|-1}}$, and in $\mathscr{F}_{\omega^\omega}$ when alphabet is not fixed [Cichon,Schmitz& S.]. Applies e.g. to lossy channel systems.

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Bottom line: one can provide definite complexity upper bounds for WQO-based algorithms

Some research goals: more varied/complex wqos (powerset, restricted families of graphs, ..) & analysis of complex algorithms

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Part 3b

Complexity: Lower Bounds

What About Lower Bounds?

Q. Are the upper bounds for Termination and Coverability optimal?

In the case of broadcast protocols:

The upper bound is tight for the algorithms we presented

But there may exist better algorithms (as with VASS, e.g.)

One can prove that the Termination and Coverability problems are $\mathscr{F}_\omega$-hard, hence $\mathscr{F}_\omega$-complete, for broadcast protocols [Urquhart,..]

and $\mathscr{F}_{\omega^\omega}$-complete for lossy channel systems [ChambartS'08],
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These results/characterizations have applications outside verification: WSTS models are often used for decidability (or hardness) of problems in logic.

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Proving F_α-Hardness

The four hardness results we just mentioned have all been proved using the same techniques:

One shows how the WSTS model can weakly compute F_α and its inverse F_α^{-1}. (Recall: broadcast protocol computing tower function)

Encode initial ordinals in $(S, \leqslant)$ & implement Hardy computations in $\mathcal{S}$.

Hardy computations: $(\alpha + 1, x) \mapsto (\alpha, x + 1)$ and $(\lambda, x) \mapsto (\lambda_x, x)$.

Main technical issue: robustness

— One easily guarantee $s \leqslant t \Rightarrow \alpha(s) \leqslant \alpha(t)$ but this does not guarantee $F_{\alpha(s)}(x) \leqslant F_{\alpha(t)}(x)$ required for weak computation of F_α.

— We need $s \leqslant t \Rightarrow \alpha(s) \sqsubseteq \alpha(t)$, using an ad-hoc stronger relation $\alpha \sqsubseteq \beta$ that entails $F_\alpha(x) \leqslant F_\beta(x)$.

PROVING F_α-HARDNESS

The four hardness results we just mentioned have all been proved using the same techniques:

One shows how the WSTS model can weakly compute F_α and its inverse F_α^{-1}. (Recall: broadcast protocol computing tower function)

Encode initial ordinals in $(S, \leqslant)$ & implement Hardy computations in $\mathcal{S}$.
Hardy computations: $(\alpha + 1, x) \mapsto (\alpha, x + 1)$ and $(\lambda, x) \mapsto (\lambda_x, x)$.

Main technical issue: robustness

— One easily guarantee $s \leqslant t \Rightarrow \alpha(s) \leqslant \alpha(t)$ but this does not guarantee $F_{\alpha(s)}(x) \leqslant F_{\alpha(t)}(x)$ required for weak computation of F_α.

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Concluding remarks

- Executive Summary

Complexity analysis of WSTS models is possible

We have complexity classes, generic techniques for upper bounds, catalog of $\mathscr{F}_\alpha$-complete problems, see S. Schmitz. Complexity hierarchies beyond Elementary. ACM Trans. Computation Theory, 8(1), 2016.

Many applications in verification and logic

- Perspectives

Need more length function theorems

There are many models for which complexity has not been narrowed

Would love to have alternative to Hardy computations …