Unary Patterns of Size Four with Morphic Permutations

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EXAMPLE

- $x = 0112$
- $\pi : 0 \to 1, 1 \to 2, 2 \to 0,$
- $x\pi(x)x\pi(x) = \underbrace{0112}_{x}\ \underbrace{1220}_{\pi(x)}\ \underbrace{0112}_{x}\ \underbrace{1220}_{\pi(x)}$

Avoidability classics

A word w *avoids* a pattern P, if it has no instance of P as a factor.

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Thue Morse word

- $0 \rightarrow 01$ and $1 \rightarrow 10$,
  $\mathbf{t} = 0110100110010110\ldots$
- Thue Morse word avoids cubes (xxx)

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Hall word (studied by Thue and Hall)

- $0 \to 012$, $1 \to 02$, and $2 \to 1$,
  $\mathbf{h} = 012021012102012021\ldots$
- Hall word avoids squares (xx)

- Bischoff, Nowotka: Pattern Avoidability with Involution (2011)
- Currie: Pattern Avoidance with Involution (2011)
- Bischoff, Currie, Nowotka: Unary patterns with involution (2012)
- Chiniforooshan, L. Kari, Xu: Pseudopower Avoidance (2012)
- Manea, Müller, Nowotka: Cubic patterns with permutations (2012)
- Currie, Manea, Nowotka, Reshadi: Unary patterns with permutations (2018)

Cubic patterns with permutations

Patterns: *cubes under permutations*, i.e., $x\pi^i(x)\pi^j(x)$.

Theorem

Given the pattern $x\pi^i(x)\pi^j(x)$ *and the type of* π *(morphic or antimorphic), we can determine all values* m *such that the pattern is avoidable in* Σ_m*.*

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Theorem

Given the pattern $x\pi^i(x)\pi^j(x)$ and the type of π (morphic or antimorphic), we can determine all values m such that the pattern is avoidable in Σ_m.

- i and j are given,
- x can be any factor, and π can be any permutation applied on x
- We want to find an alphabet m such that the pattern $x\pi^i(x)\pi^j(x)$, $i \neq j$, is unavoidable in Σ_m, for $m \geq k$
- The pattern is avoidable in in Σ_m, for $m < k$

Manea, Müller, Nowotka: The avoidability of cubes under permutations, (2012)

Avoidability of longer patterns

Example

- Consider $x\pi^3(x)\pi^5(x)$
- It is avoidable on $\Sigma = 2, 3$
- It becomes unavoidable from $\Sigma = 4$

Patterns of size four with permutations

We try to compute an interval such that all patterns $x\pi^i(x)\pi^j(x)\pi^k(x)$, with $i, j, k \geq 0$, are unavoidable.

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We try to compute an interval such that all patterns $x\pi^i(x)\pi^j(x)\pi^k(x)$, with $i, j, k \geq 0$, are unavoidable.

- pattern of size four is avoidable in Σ_2, Σ_3, Σ_4, but there exists a pattern which is unavoidable in Σ_5 ✓
- complete characterisation of the avoidability of patterns of size four with permutations

Cubic patterns with permutations

Given i and j, consider the pattern $x\pi^i(x)\pi^j(x)$. We need to define the following values:

$$
\begin{aligned}
k_1 &= \inf\{t : t \nmid |i-j|, t \nmid i, t \nmid j\} \\
k_2 &= \inf\{t : t \mid |i-j|, t \nmid i, t \nmid j\} \\
k_3 &= \inf\{t : t \mid i, t \nmid j\} \\
k_4 &= \inf\{t : t \nmid i, t \mid j\} \\
k &= \min\{\max\{k_1, k_2\}, \max\{k_1, k_3\}, \max\{k_1, k_4\}\}
\end{aligned}
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Theorem

The pattern $x\pi^i(x)\pi^j(x)$, $i \neq j$, is unavoidable in Σ_m, for $m \geq k$.

What are k_1, k_2, k_3, k_4?

- $k_1 = \inf\{t : t \nmid |i - j|, t \nmid i, t \nmid j\}$ is minimum alphabet that is needed to model the pattern $x\pi^i(x)\pi^j(x)$, with $i \neq j$, where x, $\pi^i(x)$, $\pi^j(x)$ are not similar together.
- $k_1 = \inf\{t : t \nmid |i - j|, t \nmid i, t \nmid j\} \Rightarrow x\pi^i(x)\pi^j(x) \Rightarrow$ **012** label

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- $k_2 = \inf\{t : t||i-j|, t \nmid i, t \nmid j\} \Rightarrow x\pi^i(x)\pi^j(x) \Rightarrow$ **011** label
- $k_3 = \inf\{t|i, t \nmid j\} \Rightarrow x\pi^i(x)\ \pi^j(x) \Rightarrow$ **001** label

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- $k_3 = \inf\{t | i, t \nmid j\} \Rightarrow x\pi^i(x)\ \pi^j(x) \Rightarrow$ **001** label
- $k_4 = \inf\{t \nmid i, t | j\} \Rightarrow x\pi^i(x)\pi^i(x) \Rightarrow$ **010** label

Cubic patterns with permutations

Consider the pattern $x, \pi^i(x), \pi^j(x)$, with $i \neq j$. We need to define the following values:

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\end{aligned}
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Theorem

The pattern $x, \pi^i(x), \pi^j(x)$, $i \neq j$, is unavoidable in Σ_m, for $m \geq k$.

Patterns of size four with permutations

$x\pi^i(x)\pi^j(x)\pi^k(x)$

$k_1 = \inf\{t : t \nmid i, t \nmid j, t \nmid k, t \nmid \lvert i-j \rvert, t \nmid \lvert i-k \rvert, t \nmid \lvert j-k \rvert\}$	0123
$k_2 = \inf\{t : t \mid i, t \nmid j, t \nmid k, t \nmid \lvert j-k \rvert\}$	0012
$k_3 = \inf\{t : t \nmid i, t \mid j, t \nmid k, , t \nmid \lvert i-k \rvert\}$	0102
$k_4 = \inf\{t : t \nmid i, t \nmid j, t \mid \lvert i-k \rvert\}$	0121
$k_5 = \inf\{t : t \nmid i, t \nmid j, t \nmid \lvert i-j \rvert, t \nmid \lvert i-k \rvert, t \mid \lvert j-k \rvert\}$	0122
$k_6 = \inf\{t : t \mid i, t \mid j, t \nmid k\}$	0001
$k_7 = \inf\{t : t \mid i, t \nmid j, t \mid k\}$	0010
$k_8 = \inf\{t : t \nmid i, t \mid j, t \mid k\}$	0100
$k_9 = \inf\{t : t \nmid i, t \mid \lvert i-j \rvert, t \mid \lvert i-k \rvert\}$	0111
$k_{10} = \inf\{t : t \mid i, t \nmid j, t \mid \lvert j-k \rvert\}$	0011
$k_{11} = \inf\{t : t \nmid i, t \mid j, t \mid \lvert i-k \rvert\}$	0101
$k_{12} = \inf\{t : t \nmid i, t \mid k, t \mid \lvert i-j \rvert\}$	0110
$k_{13} = \inf\{t : t \nmid i, t \nmid k, t \mid \lvert i-j \rvert\}$	0112
$k_{14} = \inf\{t : t \nmid i, t \nmid j, t \mid \lvert i-j \rvert\}$	0120

Patterns of size four with permutations

Lemma

The pattern $x\pi^i(x)\pi^j(x)\pi^k(x)$, *with* $i \neq j \neq k \neq i$ *is unavoidable in* Σ_m, *for* $m > \sigma$.

$\sigma = \min\{\max\{k_1, k_2, k_3, k_6, k_7\}, \{k_1, k_2, k_3, k_6, k_8\}, \{k_1, k_2, k_3, k_7, k_9\}, \{k_1, k_4, k_5, k_6, k_7\}, \{k_1, k_3, k_6, k_{12}, k_{13}\}, \{k_1, k_4, k_6, k_{12}, k_{13}\}, \{k_1, k_4, k_9, k_{12}, k_{13}\}, \{k_1, k_2, k_3, k_7, k_{10}\}, \{k_1, k_2, k_3, k_8, k_{10}\}, \ldots\}$

Algorithm to generate unavoidable cases

Let $\mathcal{S}_1$ be the collection of sets of 5 elements that:

- contain k_1

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- one of the k_is whose representation has a prefix or a suffix square $\Rightarrow$ ($\underline{\textbf{00}}12$, $01\underline{\textbf{22}}$) $\Rightarrow$ (k_2 or k_5)
- one of the k_is that has a gapped square

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- one of the k_is that has a gapped square ⇒ (**0**1**0**2, 0**1**2**1**) ⇒ (k_3 or k_4),

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- one of the k_is whose representation has a prefix or a suffix square $\Rightarrow$ (0012, 0122) $\Rightarrow$ (k_2 or k_5)
- one of the k_is that has a gapped square $\Rightarrow$ (0102, 0121) $\Rightarrow$ (k_3 or k_4),
- one of the k_is that contain cubes or two squares $\Rightarrow$ (0001, 0111, 0011) $\Rightarrow$ (k_6 or k_9 or k_{10}),

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- one of the k_is that contain cubes or two squares $\Rightarrow$ (**000**1, 0**111**, 00**11**) $\Rightarrow$ (k_6 or k_9 or k_{10}),
- one of the k_is that contain gapped cubes $\Rightarrow$ (**00**1**0**, **0**1**00**) $\Rightarrow$ (k_7 or k_8).

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- one of the k_is that contain cubes or two squares $\Rightarrow$ (**000**1, 0**111**, **00**~~**11**~~) $\Rightarrow$ (k_6 or k_9 or k_{10}),
- one of the k_is that contain gapped cubes $\Rightarrow$ (**00**1**0**, **0**1**00**) $\Rightarrow$ (k_7 or k_8).
- $\mathcal{S}_1 = \{\{k_1, k_2, k_3, k_6, k_7\}, \{k_1, k_2, k_3, k_6, k_8\}, \ldots$

Algorithm to generate unavoidable cases

- $\mathcal{S}_2 = \{\{k_1, k_3, k_6, k_{12}, k_{13}\}, \{k_1, k_4, k_6, k_{12}, k_{13}\}, \ldots\}$
- $\mathcal{S}_3 = \{\{k_1, k_2, k_3, k_7, k_{10}\}, \{k_1, k_2, k_3, k_8, k_{10}\}, \ldots\}$
- $\ldots$
- $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8 \cup \mathcal{S}_9 \cup \mathcal{S}_{10}$

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- $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8 \cup \mathcal{S}_9 \cup \mathcal{S}_{10}$
- Let $\sigma = \min\{\max(S) \mid S \in \cup_{1 \le \ell \le 10} S_\ell\}$. Then pattern of size four is unavoidable in Σ_m, for all $\underline{m > \sigma}$.
- The pattern of size four is avoidable for all $m \le \sigma - 1$, and becomes unavoidable on $m > \sigma$.

Algorithm to generate avoidable cases

If a set of pattern is unavoidable, all supersets of this set is unavoidable too.

Example

- $\{k_1, k_2, k_3, k_6, k_7\}$ is unavoidable set of pattern

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All subsets of avoidable sets patterns are avoidable too.

Example

- The set $\{k_1, k_2, k_5, k_6, k_8, k_{14}\}$ can be avoided by a word $\mathbf{w}$

Algorithm to generate avoidable cases

All subsets of avoidable sets patterns are avoidable too.

Example

- The set $\{k_1, k_2, k_5, k_6, k_8, k_{14}\}$ can be avoided by a word $\mathbf{w}$
- The set $\{k_1, k_2, k_5\}$ can also be avoided by $\mathbf{w}$ too

About 1400 avoidable sets of patterns will be generated.

Algorithm to generate avoidable cases

Algorithm 1

1: Let $n = 10$. Using the sets $\mathcal{S}_i, (1 \leq i \leq 10)$, generate all sets of α_is of cardinality n, that have no unavoidable sets of patterns as subset; show that they are avoidable;

2: For all n from 9 down to 4, generate all sets of cardinality n that have no unavoidable sets of patterns as subset; these sets should not be subsets of the avoidable sets of α_is of cardinality $n + 1$ (to avoid generating repetitive avoidable sets of cases generated in the past step); show that they are avoidable.

Future work

Using SAT solvers and Minizinc, find complete characterisation of the avoidability of patterns of size four with permutations, and solve avoiadablity problem for patterns with any length.

Thank you!