Abelian Anti-Powers in Infinite Words

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Regularities in Combinatorics

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We consider a different point of view, in which we look for diversity. That is, definitions of regularity based on all-distinct objects.

This point of view was introduced by G. Fici, A. Restivo, M. Silva and L. Q. Zamboni in *Anti-powers in infinite words*, J. Comb. Theory, Ser. A, 2018].

Unavoidable Regularities

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Example (Thue, 1906)

The word

$$w = 2102012101202102012021012 1\cdots$$

obtained as the fixed point of the substitution $0 \mapsto 1, 1 \mapsto 20, 2 \mapsto 210$, does not contain any square, that is, a pattern of the form xx.

Anti-powers

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Definition

A power of order n is a pattern of the form x^n (e.g., a square if $n = 2$).

An anti-power of order n is a pattern of the form $x_1 x_2 \cdots x_n$ where the x_i have the same length and are all distinct.

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Theorem

Every infinite word contains powers of any order or anti-powers of any order.

Abelian anti-powers

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We then introduce the notion of *abelian anti-powers* which is the abelian version of anti-powers:

Definition

An abelian power of order n is a pattern of the form $u_1 u_2 ... u_n$ where for every i, u_i is a permutation of u_1 (which we will denote $u_i \sim_{ab} u_1$).

An abelian anti-power of order n is a pattern of the form $u_1 u_2 \cdots u_n$ where the u_i have the same length, and for all (i, j), $u_i \nsim_{ab} u_j$.

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Definition

Suppose $\mathbb{A}$ is a n-letter alphabet, and u is a finite word on $\mathbb{A}$. Then the Parikh vector of u, noted $\Psi(u)$, is the vector of frequencies of the letters of $\mathbb{A}$ in u:

$$\Psi(u) = (|u|_{a_1}, ..., |u|_{a_n}).$$

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We can then rewrite the previous definitions:

Definition

An abelian power of order n is a pattern of the form $u_1u_2...u_n$ where for every i, $\Psi(u_i) = \Psi(u_1)$.

An abelian anti-power of order n is a pattern of the form $u_1u_2\cdots u_n$ where the u_i have the same length, and $\forall(i,j), \Psi(u_i) \neq \Psi(u_j)$.

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Definition

The *Sierpiǹski word* s is the fixed point starting with a of the substitution

$$\sigma : a \to aba$$
$$b \to bbb$$

So s begins as follows : $ababbbababbbbbbbbbababbbabab^{27}a\cdots$

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The Sierpiǹski word (whose abelian complexity grows logarithmically) does not contain abelian 11*–anti-powers*

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There is a class of words with linear abelian complexity which we were able to show contain abelian anti-powers of any order: *paperfolding words*.

In a recent article, Steȁpan Holub proved that these words contain abelian powers of any order

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Definition

Let $p_0 = (0?1?)^\omega$ and, for every $n \geq 0$, p_n as the word obtained from p_{n-1} by replacing the symbols $?$ with the letters of p_0 (with $p_{-1} = ?^\omega$). Then

$$p_0 = 0?1?0?1?0?1?0?1?0?1?0?1?0?1?0?1?\cdots,$$
$$p_1 = 001?011?001?011?001?011?001?\cdots,$$
$$p_2 = 0010011?0011011?0010011?0011\cdots,$$
$$p_3 = 001001100011011?001001110011\cdots,$$

etc. Then, $\mathbf{p} = \lim_{n\to\infty} p_n$ is the regular paperfolding word.

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Consider $\overline{p_0} = (1?0?)^\omega$.

At each step, one can decide to replace the $?$ by p_0 or by $\overline{p_0}$. This correspond to folding to the right or to the left.

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Definition

Let $\mathbf{b} \in \{0,1\}^{\mathbb{N}}$ be the sequence of instructions. For $n \in \mathbb{Z}^+$, let p_{b_n} be obtained by replacing the symbols $?$ of $p_{b_{n-1}}$ by p_0 if $b_n = 1$ and $\overline{p_0}$ if $b_n = 0$. Then

$$\mathbf{p}_b = \lim_{n\to\infty} p_{b_n}$$

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Consider a paperfolding word and two of its factors, subdivided in m subfactors of equal lenght (we will say m-*factors)*. Write $(u_1, ..., u_m)$ and $(v_1, ..., v_m)$ the m-uplets of the corresponding Parikh vectors.

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This "additivity" lemma says that, under some conditions, one can find a m-factor whose m-uplet of Parikh vectors is the sum of $(u_1, ..., u_m)$ and $(v_1, ..., v_m)$ (up to something constant).

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We first find, using the Toeplitz construction, factors of the paperfolding word of the form $w1w$ of lenght $2^k - 1 > m$. This allows us to create m m-factors whose uplets of Parikh vectors are pairwise distinct.

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We first find, using the Toeplitz construction, factors of the paperfolding word of the form $w1w$ of lenght $2^k - 1 > m$. This allows us to create m m-factors whose uplets of Parikh vectors are pairwise distinct.

Then, using Holub's lemma, by adding some multiples of their Parikh vectors it is possible to construct a m-factor filled with pairwise distinct components.

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Some more precautions have to be taken to match the conditions of Holub's lemma, although in the case of the regular paperfolding word no effort is required.

This proof allowed us to exhibit a class of words that contain both abelian powers and anti-powers of any order.

Conjecture

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Every infinite word contains abelian powers or abelian anti-powers of any order.

Conclusion

We introduced the notion of abelian anti-power, an extension of the anti-powers defined by Fici et al.

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In the future, it would be nice to extend the non-abelian theorem of unavoidable regularity in the abelian setting. Computer experimentations gave us the belief that it is still true.

Thank you