Repetition avoidance in products of factors

Pamela Fleischmann[1], Pascal Ochem[2], and Kamellia Reshadi[1]

[1] Kiel
[2] Montpellier

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Dejean

- Alphabet: $\Sigma_k = \{0, 1, \ldots, k-1\}$.
- A repetition is a word $r = pe$ such that e is a prefix of r.
- period: $|p|$.
- exponent: $\frac{|r|}{|p|} = 1 + \frac{|e|}{|p|}$.
- critical exponent: maximal exponent over the factors.
- Repetition threshold $\mathrm{RT}(k)$: minimal critical exponent over Σ_k^ω.
  - $\mathrm{RT}(2) = 2$
  - $\mathrm{RT}(3) = \frac{7}{4}$
  - $\mathrm{RT}(4) = \frac{7}{5}$
  - $\mathrm{RT}(k) = \frac{k}{k-1}$, for $k \geqslant 5$

i-terms products

- Notion introduced by Mousavi and Shallit (DLT 2013).
- $i \geqslant 1$ is an integer.
- An *i-terms product* of a word w is a word obtained by concatening i factors of w (arbitrary, not necessarily distinct).
- Example: $ississississmis = (ississ)(ississ)(mis)$ is a 3-terms product of *mississipi*.
- $\mathrm{RT}_i(k)$: we consider the exponent of i-terms products instead of factors.
- $\mathrm{RT}_1(k) = RT(k)$.

Repetition threshold for circular factors

- Notion introduced by Mousavi and Shallit (DLT 2013).
- $RTC(k)$: we consider the exponent of conjugates of factors.
- $\mathrm{RT}_2(k) = RTC(k)$.
  - Mousavi and Shallit noticed that both notions coincide for recurrent words.
  - The properties define factorial languages.
  - Every minimal subshift is uniformly recurrent.

Values of $\mathrm{RT}_i(k)$

- $\mathrm{RT}_i(2) = 2i$ $\quad (\mathrm{RT}_1(2) = 2)$
- $\mathrm{RT}_1(3) = \frac{7}{4}$
- $\mathrm{RT}_i(3) = \frac{3i}{2} + \frac{1}{4}$ for $i \geqslant 2$, i even $\quad \left(\mathrm{RT}_2(3) = \frac{13}{4}\right)$
- $\mathrm{RT}_i(3) = \frac{3i}{2} + \frac{1}{6}$ for $i \geqslant 3$, i odd
- $\mathrm{RT}_1(4) = \frac{7}{5}$
- $\mathrm{RT}_i(4) = i + \frac{1}{2}$ for $i \geqslant 2$
- $\mathrm{RT}_1(5) = \frac{5}{4}$
- $\mathrm{RT}_i(5) = i + \frac{13}{46}$ for $i \geqslant 2$
- $\mathrm{RT}_i(k) = i + \frac{1}{k-1}$ for $i \geqslant 1$ and $k \geqslant 6$ $\quad \left(\mathrm{RT}_1(k) = \frac{k}{k-1}\right)$

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$\mathrm{RT}_i(2) = 2i$

- $\mathrm{RT}_i(2) \geqslant 2i$: because squares are 2-unavoidable.
- $\mathrm{RT}_i(2) \leqslant 2i$: because of the Thue-morse word ($0 \mapsto 01$, $1 \mapsto 10$).

$\mathrm{RT}_2(3) = \frac{13}{4}$

$$
\begin{aligned}
0 &\mapsto 0435 \\
1 &\mapsto 2341 \\
2 &\mapsto 3542 \\
3 &\mapsto 3540 \\
4 &\mapsto 4134 \\
5 &\mapsto 4105
\end{aligned}
$$

$$
\begin{aligned}
0 &\mapsto 012102120102012 \\
1 &\mapsto 201020121012021 \\
2 &\mapsto 012102010212010 \\
3 &\mapsto 201210212021012 \\
4 &\mapsto 102120121012021 \\
5 &\mapsto 102010212021012
\end{aligned}
$$

$\mathrm{RT}_i(3) \geqslant \frac{3i}{2} + \frac{1}{4}$ for every even i

- We check that $\mathrm{RT}_2(3) \geqslant \frac{13}{4}$.
- There exists a 2-terms product $uv = t^e$ with $e \geqslant \frac{13}{4}$.
- So t^3 is also a 2-terms product.
- The i-terms product $(t^3)^{i/2-1}uv$ is of the form t^x with $x = 3\left(\frac{i}{2} - 1\right) + e \geqslant 3\left(\frac{i}{2} - 1\right) + \frac{13}{4} = \frac{3i}{2} + \frac{1}{4}$.

$\mathrm{RT}_i(3) \geqslant \frac{3i}{2} + \frac{1}{6}$ for every odd $i \geqslant 3$

- Suppose w_3 is a counterexample.
- w_3 is square-free.
- w_3 does not contain u and v such that
    - $uv = t^3$
    - $u = t^e$ with $e \geqslant \frac{3}{2} + \frac{1}{6} = \frac{5}{3}$
- since otherwise $uvuv \ldots u = t^x$ with $x \geqslant \frac{3i}{2} + \frac{1}{6}$.
- w_3 does not contain both $u = abcab$ and $v = cabc$.
- w_3 does not contain both $u = abcbabc$ and $v = babcb$.
- Backtracking shows that w_3 does not exist.

$\mathrm{RT}_i(3) \leqslant \frac{3i}{2} + \frac{1}{4}$ for every even i

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0 ↦  010201210212021012102010212012101202101210212
1 ↦  010201210212012101202101210201021202101210212
2 ↦  010201210120212012102120210121021201210120212
3 ↦  010201210120210121021201210120212012102010212
```

- Let w_3 be the image of any infinite $7/5^+$-free word over Σ_4 by this 45-uniform morphism.
- w_3 is $\left(\frac{202}{135}^+, 36\right)$-free.
- So we consider only repetitions of period at most 35.
- w_3 does not contain u and v such that
  - $uv = t^e$,
  - $e > 3$,
  - $9 \leqslant |t| \leqslant 35$.
- So we consider only repetitions of period at most 8.

$\mathrm{RT}_i(3) \leqslant \frac{3i}{2} + \frac{1}{6}$ for every odd $i \geqslant 3$

- Let w_3 be the image of any infinite $7/5^+$-free word over Σ_4 by this 514-uniform morphism.
- w_3 is $\left(\frac{3}{2}^+, 45\right)$-free.
- So we consider only repetitions of period at most 44.

$\mathrm{RT}_i(3) \leqslant \frac{3i}{2} + \frac{1}{6}$ for every odd $i \geqslant 3$

01020120210120102120210201210120102012021020121021201020121012021020121021
20210120102012102120102012021020121012010212021020121021201020120210120102
12021020121021202101201020121021201020121012021020121021202101201021202102
01210120102012021020121012010212021020121021201020121012021020121021201020
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Larger alphabets ?

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Thank you