Return words and derivated sequences to Rote sequences

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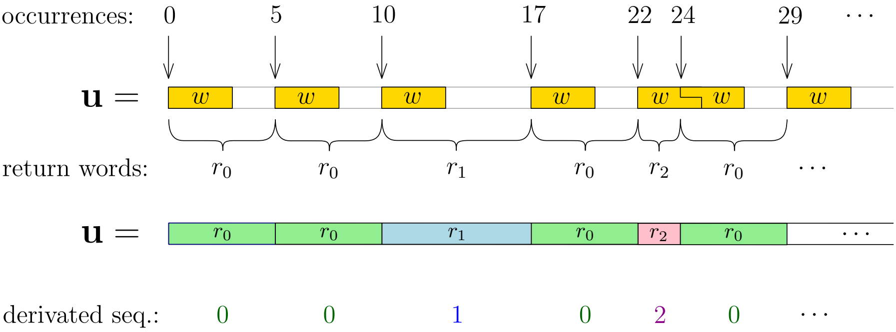
Return words and derivated sequences (Durand 1998)



A **return word** to a prefix w in $\mathbf{u}$ is the word $u_i u_{i+1} \cdots u_{j-1}$ for every two consecutive occurrences $i < j$ of w in $\mathbf{u}$.

Let $r_0, \ldots, r_{k-1}$ be return words to w in $\mathbf{u}$. Then $\mathbf{u} = r_{s_0} r_{s_1} r_{s_2} \cdots$.
The sequence $\mathbf{d_u}(w) = s_0 s_1 s_2 \cdots$ is the **derivated sequence** of $\mathbf{u}$ to w.

Complementary symmetric Rote sequences (Rote 1994)

A sequence $\mathbf{v}$ is a **Rote sequence** if $\mathcal{C}_{\mathbf{v}}(n) = 2n$ for all $n \in \mathbb{N}$.
A sequence is **complementary symmetric (CS)** if its language is closed under the exchange of letters $0 \leftrightarrow 1$.

Example: $\mathbf{v} = 0\,0\,1\,1\,1\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,0\,0\,0\,1\,1\,0\,0\,0\,1\,1\,1\,0\,0\,1\,1 \cdots$
has factors: $0, 1, 00, 01, 10, 11, 000, 001, 011, 100, 110, 111, \ldots$

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CS Rote sequences are **neutral with characteristic** 0:

- neutral: every non-empty factor w has its bilateral order $m_{\mathbf{v}}(w) = 0$;
- characteristic of $\mathbf{v} = 1 - m_{\mathbf{v}}(\varepsilon)$.

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Return words to CS Rote sequences

Theorem

Every non-empty prefix x of a CS Rote sequence $\mathbf{v}$ has exactly three return words.

It is a direct consequence of

- Balková, Pelantová, Steiner 2008
- Dolce, Perrin 2017

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To study derivated sequences we need to know also the structure of return words, not only their number.

Complementary symmetric Rote sequences

Sequence $\mathbf{v}$ is **CS Rote sequence** if $\mathcal{C}_{\mathbf{v}}(n) = 2n$ and its language is closed under the exchange of letters $0 \leftrightarrow 1$.

Theorem (Rote 1994)

A sequence $\mathbf{v} = v_0 v_1 v_2 \cdots$ is a CS Rote sequence if and only if its difference sequence $\mathbf{u} = u_0 u_1 u_2 \cdots$ which is defined by

$$u_i = v_{i+1} - v_i = v_{i+1} + v_i \mod 2$$

is a Sturmian sequence. We denote $\mathcal{S}(\mathbf{v}) = \mathbf{u}$.

$$\left.\begin{array}{ll} \text{Rote:} & \mathbf{v}_F = 0\,0\,1\,1\,1\,0\,0\,1\,1\,1\,0\,0\,0\,1\,1\,0\,0\,0\,1\,1\,0\,0\,0\,1\,1\cdots \\ \text{Sturmian:} & \mathbf{u}_F = \ 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\cdots \end{array}\right\} \mathcal{S}(\mathbf{v}_F) = \mathbf{u}_F$$

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$\mathcal{S}(00) = \mathcal{S}(11) = 0$ $\qquad$ $\mathcal{S}(001) = 01$ $\qquad$ $\mathcal{S}(001110) = 01001$

Complementary symmetric Rote sequences

Sturmian: $\mathbf{u}_F = 01001010010010100101001001010\cdots$

Rote: $\mathbf{v}_F = 0\,00111001110001100011000111001 1\cdots$

Rote: $\mathbf{v}_F = 1\,11000110001110011100111000110 0\cdots$

Return words to CS Rote sequences

Sturmian: $\mathbf{u}_F = \ 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0 \cdots$

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We have to decide when 00 and 11 appear.

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Strumian: $\mathbf{u}_F = \ 0\,1 \mid 0 \mid 0\,1 \mid 0\,1 \mid 0 \mid 0\,1 \mid 0 \mid 0\,1 \mid 0\,1 \mid 0 \mid 0\,1 \mid 0\,1 \mid 0 \mid 0\,1 \mid 0 \mid 0\,1 \mid 0\,1 \mid 0\cdots$

Rote: $\mathbf{v}_F = 0\,0\,1\,1\,1 \mid 0\,0\,1\,1\,1 \mid 0 \mid 0\,0\,1\,1 \mid 0 \mid 0\,0\,1\,1 \mid 0 \mid 0\,0\,1\,1\,1 \mid 0\,0\,1\,1\cdots$

Return words to CS Rote sequences

Lemma

Let x be a prefix of $\mathbf{v}$. A number i is an occurrence of x in $\mathbf{v}$ if and only if i is an occurrence of $\mathcal{S}(x)$ in $\mathbf{u} = \mathcal{S}(\mathbf{v})$ and the number of 1's in the prefix $\mathbf{u}_{[0,i-1)}$ is even.

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A word $u = u_0 u_1 \cdots u_{n-1}$ is called **stable** (S) if $|u|_1 = 0 \mod 2$, where $|u|_1$ denotes the number of occurrences of the letter 1 in u.
Otherwise, u is **unstable** (U).

Return words to CS Rote sequences

Let w be a prefix of a standard Sturmian sequence $\mathbf{u}$ with return words r, s and let $\mathbf{u}$ be a concatenation of blocks $r^k s$ and $r^{k+1}s$. We distinguish:

i) w is of type $SU(k)$ if r is stable and s is unstable;

ii) w is of type $US(k)$ if r is unstable and s is stable;

iii) w is of type $UU(k)$ if both r and s are unstable.

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Theorem

Let $\mathbf{v}$ be a CS Rote sequence with a prefix x and let $\mathbf{u} = \mathcal{S}(\mathbf{v})$ be its Sturmian sequence with the prefix $w = \mathcal{S}(x)$ and its return words r, s. Then the return words A, B, C to x satisfy

i) if w is $SU(k)$: $r = \mathcal{S}(A0)$, $sr^{k+1}s = \mathcal{S}(B0)$, $sr^k s = \mathcal{S}(C0)$;

ii) if w is $US(k)$: $rr = \mathcal{S}(A0)$, $rsr = \mathcal{S}(B0)$, $s = \mathcal{S}(C0)$;

iii) if w is $UU(k)$: $rr = \mathcal{S}(A0)$, $rs = \mathcal{S}(B0)$, $sr = \mathcal{S}(C0)$.

Derivated sequences to CS Rote sequences

Corollary

Let $\mathbf{v}$ be a CS Rote sequence with a non-empty prefix x, $\mathbf{u} = \mathcal{S}(\mathbf{v})$ be a standard Sturmian sequence. Then the derivated sequence $\mathbf{d_v}(x)$ is uniquely determined by

i) $\mathbf{d_u}(w)$ to the prefix $w = \mathcal{S}(x)$ in $\mathbf{u}$ and

ii) the type of w.

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i) the derivated sequences of Sturmian sequences are known

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i) the derivated sequences of Sturmian sequences are known

ii) the types are determined by S-adic representations of Sturmian sequences

Derivated sequences to CS Rote sequences

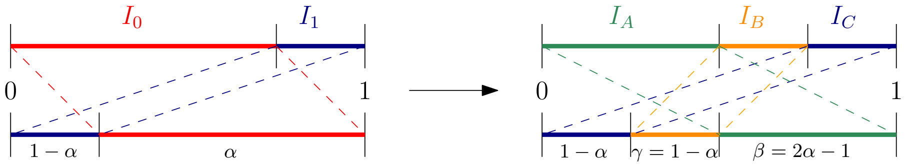
Proposition

Let **v** be a CS Rote seq., $\mathbf{u} = \mathcal{S}(\mathbf{v})$ be a standard Sturmian seq. Let x be a non-empty prefix of **v** and $w = \mathcal{S}(x)$. Let $\alpha > \frac{1}{2}$ be the slope of $\mathbf{d_u}(w)$. Then $\mathbf{d_v}(x)$ is a coding of 3iet transformation given by lengths $\beta, \gamma, 1 - \beta - \gamma$ and by permutation π, where

i) if w is $SU(k)$, $\beta = \alpha$, $\gamma = \alpha - k(1-\alpha)$ and $\pi = (3, 2, 1)$;

ii) if w is $US(k)$, $\beta = 2\alpha - 1$, $\gamma = 1 - \alpha$ and $\pi = (3, 2, 1)$;

iii) if w is $UU(k)$, $\beta = 2\alpha - 1$, $\gamma = 1 - \alpha$ and $\pi = (2, 3, 1)$.



Primitive substitutive CS Rote sequences

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A sequence is substitutive primitive if and only if it has finitely many derivated sequences.

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Proposition

Let $\mathbf{v}$ be a CS Rote sequence, $\mathbf{u} = \mathcal{S}(\mathbf{v})$ be a standard Sturmian sequence. Then $\mathbf{v}$ is not a fixed point of primitive morphism.

Derivated sequences to CS Rote sequences

Corollary

Let $\mathbf{v}$ be a CS Rote seq. and let $\mathbf{u} = \mathcal{S}(\mathbf{v})$ be a standard Sturmian seq. which is a fixed point of a primitive morphism. If $\mathbf{u}$ has K derivated sequences, then $\mathbf{v}$ has at most $3K$ derivated sequences and each of them is fixed by a primitive morphism.

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Let $\mathbf{v}$ be a CS Rote seq. and let $\mathbf{u} = \mathcal{S}(\mathbf{v})$ be a standard Sturmian seq. which is a fixed point of a primitive morphism. If $\mathbf{u}$ has K derivated sequences, then $\mathbf{v}$ has at most $3K$ derivated sequences and each of them is fixed by a primitive morphism.

$\mathbf{v}_F$ has three derivated seq. fixed by the morphisms:

$$\sigma_0 : \begin{cases} A \to AB \\ B \to ABAACAACA \\ C \to ABAACA \end{cases} \quad \sigma_1 : \begin{cases} A \to BBCAC \\ B \to BBCACAC \\ C \to B \end{cases} \quad \sigma_2 : \begin{cases} A \to BACCB \\ B \to BACC \\ C \to BACB \end{cases}$$

Derivated sequences to CS Rote sequences

$\mathbf{u}$ is a fixed point of $0 \to 101$, $1 \to 10$

the associated Rote seq. $\mathbf{v}$ has two derivated seq. fixed by:

$$\sigma_0 : \begin{cases} A \to ABCB \\ B \to A \\ C \to AB \end{cases} \qquad \sigma_1 : \begin{cases} A \to ABC \\ B \to AC \\ C \to AB \end{cases}$$

Comments

- We focused on standard Sturmian sequences.
- CS Rote sequences are neutral sequences of characteristics 0, but they are not dendric.
- The derivated sequences of CS Rote sequences are 3iet sequences and so they are dendric.
- A CS Rote sequence is primitive substitutive if and only if its associated Sturmian sequence is primitive substitutive.
- We can list the morphisms which fix the derivated sequences of a given CS Rote sequence if its associated Sturmian sequence is a fixed point.

Thank you for your attention!