Avoiding additive powers - Algorithmic proofs

Florian Lietard

Supervisors : Damien Jamet (LORIA) and Thomas Stoll (IECL)

Powers

An infinite word $w = 31410421030342432334121432132 14 \cdots$

Powers

An infinite word $w = 31410421030342432334121432 13214\cdots$

w contains a (pure) square : same blocks

$$w = 3141042103034243233341214 \cdot 321 \cdot 321 \cdot 4 \cdots$$

Powers

An infinite word $w = 31410421030342432334121432 13214\cdots$

w contains a (pure) square : same blocks

$$w = 3141042103034243233412 14 \cdot 321 \cdot 321 \cdot 4 \cdots$$

w contains an abelian square : same blocks up to a permutation

$$w = 31410421030 \cdot 342 \cdot 432 \cdot 33412143213214 \cdots$$

Powers

An infinite word $w = 31410421030342432334121432132 14\cdots$

w contains a (pure) square : same blocks

$$w = 31410421030342432334 1214 \cdot 321 \cdot 321 \cdot 4\cdots$$

w contains an abelian square : same blocks up to a permutation

$$w = 31410421030 \cdot 342 \cdot 432 \cdot 33412143213214\cdots$$

w contains an additive square : same size and same sum

$$w = 31 \cdot \underbrace{410421}_{\Sigma=12} \cdot \underbrace{030342}_{\Sigma=12} \cdot 4323341214321 3214\cdots$$

Powers

An infinite word $w = 31410421030342432334121432132 14\cdots$

w contains a (pure) square : same blocks

$$w = 31410421030342432334 1214 \cdot 321 \cdot 321 \cdot 4\cdots$$

w contains an abelian square : same blocks up to a permutation

$$w = 31410421030 \cdot 342 \cdot 432 \cdot 33412143213214\cdots$$

w contains an additive square : same size and same sum

$$w = 31 \cdot \underbrace{410421}_{\Sigma=12} \cdot \underbrace{030342}_{\Sigma=12} \cdot 43233412143213214\cdots$$

These notions can naturally be extended to higher powers, such as cubes ...

Avoidability

Objective

Construct infinite words over finite alphabets avoiding such patterns

Avoidability

Objective

Construct infinite words over finite alphabets avoiding such patterns

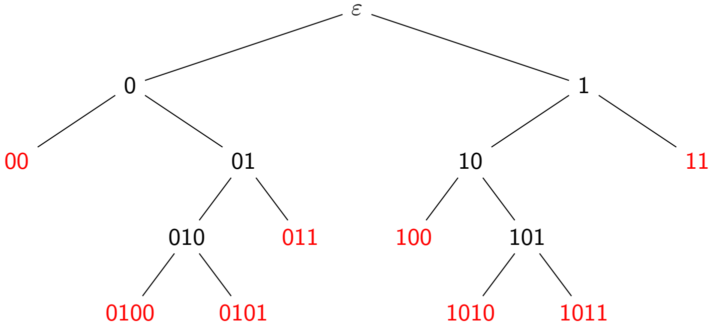
All words of size ≥ 4 over $\{0, 1\}$ contain squares

Avoidability

Objective

Construct infinite words over finite alphabets avoiding such patterns

All words of size ≥ 4 over $\{0, 1\}$ contain squares



But it is possible over $\{0, 1, 2\}$ (A. Thue, 1912)

Context and motivations

Uniformly k-repetitive semigroups

A semigroup S is **uniformly-k-repetitive** if for all morphisms $\varphi : \Sigma^+ \to S$ and for all words $w \in \Sigma^+$ long enough, there exists a factor $w_1 \cdots w_k$ in w such that

$$\varphi(w_1) = \cdots = \varphi(w_k) \text{ and } |w_1| = \cdots = |w_k|$$

Context and motivations

Uniformly k-repetitive semigroups

A semigroup S is **uniformly-k-repetitive** if for all morphisms $\varphi : \Sigma^+ \to S$ and for all words $w \in \Sigma^+$ long enough, there exists a factor $w_1 \cdots w_k$ in w such that

$$\varphi(w_1) = \cdots = \varphi(w_k) \text{ and } |w_1| = \cdots = |w_k|$$

Question of Pirillo and Varricchio (1994)

Is $\mathbb{N}^+$ uniformly k-repetitive for $k \geq 2$?

Context and motivations

Uniformly k-repetitive semigroups

A semigroup S is **uniformly-k-repetitive** if for all morphisms $\varphi : \Sigma^+ \to S$ and for all words $w \in \Sigma^+$ long enough, there exists a factor $w_1 \cdots w_k$ in w such that

$$\varphi(w_1) = \cdots = \varphi(w_k) \text{ and } |w_1| = \cdots = |w_k|$$

Question of Pirillo and Varricchio (1994)

Is $\mathbb{N}^+$ uniformly k-repetitive for $k \geq 2$?

Partial answer (J. Cassaigne *et al.*)

$\mathbb{N}^+$ is not uniformly 3-repetitive

J. Justin, 1972
Généralisation du théorème de Van der Waerden sur les semi-groupes répétitifs,
In *Journal of combinatorial theory (A)*, Volume 12, 357-367, 1972.

G. Pirillo, S. Varricchio, 1994
On uniformly repetitive semigroups,
In *Semigroup Forum*, Volume 49, 125-129, 1994.

State of the art

Problem : Find an infinite word avoiding pure/abelian/additive powers

	Pure	**Abelian**	**Additive**	
cubes	2 letters 1906	3 letters 1979	4 letters 2014	3 letters 2015
squares	3 letters 1912	4 letters 1992	?	

1906 - A.Thue
Über unendliche Zeichenreihen,
Skrifter udgivne af Videnskabsselskabet i Christiania : Mathematisk-naturvidenskabelig Klasse, 1-22, 1906

1912 - A.Thue
Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen,
Skrifter udgivne af Videnskabsselskabet i Christiania : Mathematisk-naturvidenskabelig Klasse, 1-67, 1912

1979 - F.M. Dekking
Strongly non-repetitive sequences and progression-free sets,
In *Journal of Combinatorial Theory, Series A*, Volume 27, 181-185, 1979

1992 - V. Keränen
Abelian squares are avoidable on 4 letters,
In *Automata, Languages and Programming*, July 13 – 17, 41-52, 1992

2014 - J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit
Avoiding Three Consecutive Blocks of the Same Size and Same Sum,
In *Journal of the ACM* , Volume 61, issue no.2, April 2014

2015 - M. Rao
On some generalizations of abelian power avoidability,
In *Theoretical Computer Science*, (601) 39-46, 2015

State of the art

A 4-letter morphism avoiding additive cubes [J. Cassaigne *et* al. 2014]

$$\varphi_0 : 0 \mapsto 03, \quad 1 \mapsto 43, \quad 3 \mapsto 1, \quad 4 \mapsto 01$$

$$\varphi_0^\infty(0) = 031430110343430310110110314303434303434303434301\cdots$$

J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit (2014)
Avoiding Three Consecutive Blocks of the Same Size and Same Sum,
In *Journal of the ACM* , Volume 61, issue no.2, April 2014

It is possible to avoid additive cubes over a 4-letter alphabet with a morphism of size 2

State of the art

A 4-letter morphism avoiding additive cubes [J. Cassaigne *et* al. 2014]

$$\varphi_0 : 0 \mapsto 03, \qquad 1 \mapsto 43, \qquad 3 \mapsto 1, \qquad 4 \mapsto 01$$

$$\varphi_0^{\infty}(0) = 03143011034343031011011031430343430343430314301 \cdots$$

J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit (2014)
Avoiding Three Consecutive Blocks of the Same Size and Same Sum,
In *Journal of the ACM* , Volume 61, issue no.2, April 2014

It is possible to avoid additive cubes over a 4-letter alphabet with a morphism of size 2

State of the art

A 4-letter morphism avoiding additive cubes [J. Cassaigne *et* al. 2014]

$$\varphi_0 : 0 \mapsto 03, \quad 1 \mapsto 43, \quad 3 \mapsto 1, \quad 4 \mapsto 01$$

$$\varphi_0^\infty(0) = 031430110343430310110110314303434303434303 14301\cdots$$

J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit (2014)
Avoiding Three Consecutive Blocks of the Same Size and Same Sum,
In *Journal of the ACM* , Volume 61, issue no.2, April 2014

It is possible to avoid additive cubes over a 4-letter alphabet with a morphism of size 2

State of the art

A 4-letter morphism avoiding additive cubes [J. Cassaigne *et* al. 2014]

$$\varphi_0 : 0 \mapsto 03, \quad 1 \mapsto 43, \quad 3 \mapsto 1, \quad 4 \mapsto 01$$

$$\varphi_0^\infty(0) = 031430110343430310110110314303434303434303 14301 \cdots$$

J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit (2014)
Avoiding Three Consecutive Blocks of the Same Size and Same Sum,
In *Journal of the ACM* , Volume 61, issue no.2, April 2014

It is possible to avoid additive cubes over a 4-letter alphabet with a morphism of size 2

Questions

	Pure	Abelian	Additive	
cubes	2 letters 1906	3 letters 1979	4 letters 2014	3 letters 2015
squares	3 letters 1912	4 letters 1992	?	

Do there exist :

- many 4-letter morphisms avoiding additive cubes ?
- morphic words without additive cubes but with non-abelian additive squares ?

$$\mathbf{w} = 6021062260101 \cdot \overbrace{06026}^{\Sigma=14} \cdot \overbrace{22622}^{\Sigma=14} \cdot 6021060101060101 \cdots$$

$$\mathbf{w_0} = 0314301103434 \cdot \underbrace{30310}_{\Sigma=7} \cdot \underbrace{11011}_{\Sigma=4} \cdot 0314303434303434 \cdots$$

Our approach

- Want to find other morphic words on other alphabets
- Compute to get some intuition
- In w_0 all additive squares are abelian squares : sufficient to show that w_0 avoids abelian cubes

Our approach

- Want to find other morphic words on other alphabets
- Compute to get some intuition
- In w_0 all additive squares are abelian squares : sufficient to show that w_0 avoids abelian cubes

Our experimental results

- We find 5% of morphic words with additive and non-abelian squares
- All morphisms are similar to φ_0

Apply it to other morphisms

$$\varphi_0(0) = 03 \qquad \varphi_0(1) = 43$$
$$\varphi_0(3) = 1 \qquad \varphi_0(4) = 01$$

The corresponding incidence matrix :

$$\mathrm{Mat}(\varphi_0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Apply it to other morphisms

$$\varphi_0(0) = 03 \qquad \varphi_0(1) = 43$$
$$\varphi_0(3) = 1 \qquad \varphi_0(4) = 01$$

The corresponding incidence matrix :

$$\mathrm{Mat}(\varphi_0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\varphi(6) = 60 \qquad \varphi(2) = 10$$
$$\varphi(0) = 2 \qquad \varphi(1) = 62$$

$$\mathbf{w} = \lim_{n\to\infty} \varphi^n(6) = 6021062260101060262262260210601010601 01\cdots$$

Apply previous proof to other morphisms

Lemma

If a morphism is similar to φ_0, then it fits the informatic proof developped by Cassaigne *et al.* in 2014.

Theorem (Jamet, L., Stoll)

Let w be a fixed point of a morphism similar to φ_0. The following propositions are decidable :

- w avoids additive cubes
- in w, all additive squares are abelian squares

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) & = 2 \\ \varphi(1) & = 62 \\ \varphi(2) & = 10 \\ \varphi(6) & = 60 \end{cases}$$

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) &= 2 \\ \varphi(1) &= 62 \\ \varphi(2) &= 10 \\ \varphi(6) &= 60 \end{cases}$$

- $\varphi^1(6) = 60$

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) & = 2 \\ \varphi(1) & = 62 \\ \varphi(2) & = 10 \\ \varphi(6) & = 60 \end{cases}$$

- $\varphi^1(6) = 60$
- $\varphi^2(6) = 602$

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) & = 2 \\ \varphi(1) & = 62 \\ \varphi(2) & = 10 \\ \varphi(6) & = 60 \end{cases}$$

- $\varphi^1(6) = 60$
- $\varphi^2(6) = 602$
- $\varphi^3(6) = 60210$

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) & = 2 \\ \varphi(1) & = 62 \\ \varphi(2) & = 10 \\ \varphi(6) & = 60 \end{cases}$$

- $\varphi^1(6) = 60$
- $\varphi^2(6) = 602$
- $\varphi^3(6) = 60210$
- $\varphi^4(6) = 60210622$

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) & = 2 \\ \varphi(1) & = 62 \\ \varphi(2) & = 10 \\ \varphi(6) & = 60 \end{cases}$$

- $\varphi^1(6) = 60$
- $\varphi^2(6) = 602$
- $\varphi^3(6) = 60210$
- $\varphi^4(6) = 60210622$
- $\varphi^5(6) = 60210622601010$

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) & = 2 \\ \varphi(1) & = 62 \\ \varphi(2) & = 10 \\ \varphi(6) & = 60 \end{cases}$$

- $\varphi^1(6) = 60$
- $\varphi^2(6) = 602$
- $\varphi^3(6) = 60210$
- $\varphi^4(6) = 60210622$
- $\varphi^5(6) = 6021062260 1010$

Sketch of the proof

Why do we choose morphic words ?

$$\begin{cases} \varphi(0) & = 2 \\ \varphi(1) & = 62 \\ \varphi(2) & = 10 \\ \varphi(6) & = 60 \end{cases}$$

- $\varphi^1(6) = 60$
- $\varphi^2(6) = 602$
- $\varphi^3(6) = 60210$
- $\varphi^4(6) = 60210622$
- $\varphi^5(6) = 6021062260101 0$

$\mathbf{w} = 6021062260101\mathbf{0}60262262260210601010 60101$

$\mathbf{w}[p]$	6	0	2	1	0	6	2	2	6	0	1	0	1	**0**
p	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathbf{par}(p)$	0	0	1	2	2	3	3	4	5	5	6	6	7	7

Where the alphabet matters

Parikh vector

The Parikh vector $\psi(x)$ of a word x is :

$$\psi(x) = \begin{pmatrix} |x|_0 \\ |x|_1 \\ |x|_2 \\ |x|_6 \end{pmatrix}, \text{ example : } \psi(60210) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Where the alphabet matters

Parikh vector

The Parikh vector $\psi(x)$ of a word x is :

$$\psi(x) = \begin{pmatrix} |x|_0 \\ |x|_1 \\ |x|_2 \\ |x|_6 \end{pmatrix}, \text{ example : } \psi(60210) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

If b and c are two blocks with same length and same sum then the vector $\mathbf{v} = \psi(b) - \psi(c)$ belongs to the lattice

$$\mathfrak{L} := \{\mathbf{v} \in \mathbb{Z}^4 : (1,1,1,1) \cdot \mathbf{v} = 0 \text{ et } (0,1,2,6) \cdot \mathbf{v} = 0\}.$$

which depends on the chosen alphabet.

Linear algebra

Let p be a position, we define

$$\sigma(p) = \psi(\mathbf{w}[0,p)) = \begin{pmatrix} |\mathbf{w}[0,p)|_0 \\ |\mathbf{w}[0,p)|_1 \\ |\mathbf{w}[0,p)|_2 \\ |\mathbf{w}[0,p)|_6 \end{pmatrix}, \text{ example : } \sigma(9) = \psi(602106226) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

Linear algebra

Let p be a position, we define

$$\sigma(p) = \psi(\mathbf{w}[0,p)) = \begin{pmatrix} |\mathbf{w}[0,p)|_0 \\ |\mathbf{w}[0,p)|_1 \\ |\mathbf{w}[0,p)|_2 \\ |\mathbf{w}[0,p)|_6 \end{pmatrix}, \text{ example : } \sigma(9) = \psi(602106226) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

Lemma (J. Cassaigne *et* al., 2014)

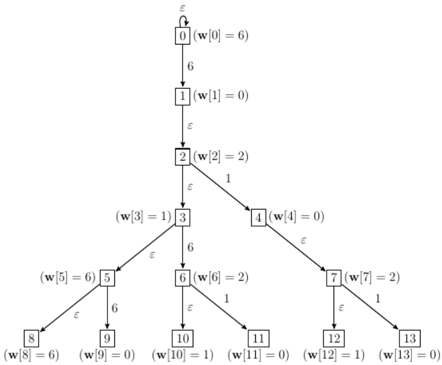
If q is a child of p and a the proper prefix linked to q (via the bijection), we get

$$\sigma(q) = M\sigma(p) + \psi(a)$$

Example :

$$\sigma(9) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \times \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{=\sigma(5)=\psi(60210)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{=\psi(6)}$$

Walk on a tree



Corollary (J. Cassaigne *et* al., 2014)

If $\{p_i\}_{i=0}^{\infty}$ is the ancestral sequence of a position p and denoting a_i the proper prefix used to link p_i to p_{i+1}, we get : $\sigma(p_0) = \sum_{i=0}^{\infty} M^i \psi(a_i)$.

So, how does it work ?

- Using parents and graphs, we get bounds for $\mathbf{v} = \psi(b) - \psi(c)$
- Using the lattice, we get other bounds for $\mathbf{v}$
- $\mathbf{v}$ lies in a ball of fixed radius
- This ball allow us to consider a finite subgraph
- We detect additive cubes by computing

So, how does it work ?

- Using parents and graphs, we get bounds for $\mathbf{v} = \psi(b) - \psi(c)$
- Using the lattice, we get other bounds for $\mathbf{v}$
- $\mathbf{v}$ lies in a ball of fixed radius
- This ball allow us to consider a finite subgraph
- We detect additive cubes by computing

Using exactly the same arguments but considering two consecutive blocks rather than three, it is possible to detect additive squares

Statistics

Morphisms of size 2

- 32068 morphisms avoiding additive cubes, over 4-letters alphabets included in $\{0, 1, \ldots, 25\}$
- Less than 5% with a fixed point containing additive non-abelian squares
- 23 morphisms avoiding additive cubes over $\{0, 1, 5, 25\}$
- 2 morphisms avoiding additive cubes over $\{0, 2, 5, 11\}$
- At least one morphism for each alphabet included in $\{0, 1, \ldots, 25\}$ except $\{0, 1, 2, 3\}$ and $\{0, 1, 2, 4\}$.
- All morphisms avoiding additive cubes are similar to φ_0

Statistics

Morphisms of size 3

- 132 morphisms over 4-letters alphabets $\{0, 1, 2, c\}$ $(4 \leq c \leq 9)$ avoiding additive cubes
- Not all similar to φ_0 : there is an other class
- 9 morphisms avoiding additive cubes over the alphabet $\{0, 1, 2, 4\}$, 5 are similar to φ_0

To be continued

Proposition (Jamet, L., Stoll)

The following morphisms avoid additive cubes :

$$\varphi_2 : \begin{cases} 0 \mapsto 21 \\ 1 \mapsto 011 \\ \underline{2} \mapsto 214 \\ 4 \mapsto 244 \end{cases} \quad \text{and} \quad \varphi_3 : \begin{cases} 0 \mapsto 4 \\ \underline{1} \mapsto 12 \\ 2 \mapsto 0 \\ 4 \mapsto 100 \end{cases}$$

Where φ_3 is similar to φ_0

To be continued

Proposition (Jamet, L., Stoll)

The following morphisms avoid additive cubes :

$$\varphi_2 : \begin{cases} 0 \mapsto 21 \\ 1 \mapsto 011 \\ \underline{2} \mapsto 214 \\ 4 \mapsto 244 \end{cases} \quad \text{and} \quad \varphi_3 : \begin{cases} 0 \mapsto 4 \\ \underline{1} \mapsto 12 \\ 2 \mapsto 0 \\ 4 \mapsto 100 \end{cases}$$

Where φ_3 is similar to φ_0

Question

If φ is a morphism avoiding additive cubes, do there exist intergers k and n such that :

$$\varphi^k \simeq \varphi_0^n$$

Thank you for your attention