Avoiding additive powers - Algorithmic proofs

Florian Lietard

Supervisors: Damien Jamet (LORIA) and Thomas Stoll (IECL)
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\( w \) contains a **(pure) square**: same blocks

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- $w$ contains an additive square: same size and same sum
  
  $w = 31 \cdot \underbrace{410421 \cdot 030342}_{\Sigma=12} \cdot 43233412143213214 \cdots$
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$\Sigma = 12$ $\Sigma = 12$

These notions can naturally be extended to higher powers, such as cubes ...
Avoidability

Objective
Construct infinite words over finite alphabets avoiding such patterns
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All words of size $\geq 4$ over $\{0, 1\}$ contain squares
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Construct infinite words over finite alphabets avoiding such patterns

All words of size $\geq 4$ over $\{0, 1\}$ contain squares

But it is possible over $\{0, 1, 2\}$ (A. Thue, 1912)
Uniformly k-repetitive semigroups

A semigroup $S$ is uniformly-k-repetitive if for all morphisms $\varphi : \Sigma^+ \to S$ and for all words $w \in \Sigma^+$ long enough, there exists a factor $w_1 \cdots w_k$ in $w$ such that

$$\varphi(w_1) = \cdots = \varphi(w_k) \text{ and } |w_1| = \cdots = |w_k|$$
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Question of Pirillo and Varricchio (1994)

Is $\mathbb{N}^+$ uniformly k-repetitive for $k \geq 2$?
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Is $\mathbb{N}^+$ uniformly k-repetitive for $k \geq 2$?

Partial answer (J. Cassaigne et al.)

$\mathbb{N}^+$ is not uniformly 3-repetitive

- J. Justin, 1972
  Généralisation du théorème de Van der Waerden sur les semi-groupes répétitifs,

- G. Pirillo, S. Varricchio, 1994
  On uniformly repetitive semigroups,
State of the art

Problem: Find an infinite word avoiding pure/abelian/additive powers

<table>
<thead>
<tr>
<th></th>
<th>Pure</th>
<th>Abelian</th>
<th>Additive</th>
</tr>
</thead>
<tbody>
<tr>
<td>cubes</td>
<td>2 letters 1906</td>
<td>3 letters 1979</td>
<td>4 letters 2014</td>
</tr>
<tr>
<td>squares</td>
<td>3 letters 1912</td>
<td>4 letters 1992</td>
<td>?</td>
</tr>
</tbody>
</table>

1906 - A. Thue
Über unendliche Zeichenreihen,
Skrifter udgivne af Videnskabsselskabet i Christiania: Mathematisk-naturvidenskabelig Klasse, 1-22, 1906

1912 - A. Thue
Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen,
Skrifter udgivne af Videnskabsselskabet i Christiania: Mathematisk-naturvidenskabelig Klasse, 1-67, 1912

1979 - F.M. Dekking
Strongly non-repetitive sequences and progression-free sets,
In Journal of Combinatorial Theory, Series A, Volume 27, 181-185, 1979

1992 - V. Keränen
Abelian squares are avoidable on 4 letters,
In Automata, Languages and Programming, July 13 – 17, 41-52, 1992

2014 - J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit
Avoiding Three Consecutive Blocks of the Same Size and Same Sum,
In Journal of the ACM, Volume 61, issue no.2, April 2014

2015 - M. Rao
On some generalizations of abelian power avoidability,
In Theoretical Computer Science, (601) 39-46, 2015
A 4-letter morphism avoiding additive cubes [J. Cassaigne et al. 2014]

\[ \varphi_0 : 0 \mapsto 03, \quad 1 \mapsto 43, \quad 3 \mapsto 1, \quad 4 \mapsto 01 \]

\[ \varphi_0^\infty (0) = 0314301103434303101101103143034343034303430314301 \cdots \]

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It is possible to avoid additive **cubes** over a 4-letter **alphabet** with a morphism of size 2
Do there exist:
- many 4-letter morphisms avoiding additive cubes?
- morphic words without additive cubes but with non-abelian additive squares?

\[
\begin{array}{|c|c|c|c|}
\hline
 & \text{Pure} & \text{Abelian} & \text{Additive} \\
\hline
\text{cubes} & 2 \text{ letters} & 3 \text{ letters} & 4 \text{ letters} \\
 & 1906 & 1979 & 2014 \\
\hline
\text{squares} & 3 \text{ letters} & 4 \text{ letters} & ? \\
 & 1912 & 1992 & \\
\hline
\end{array}
\]
Our approach

- Want to find other morphic words on other alphabets
- Compute to get some intuition
- In $w_0$ all additive squares are abelian squares: sufficient to show that $w_0$ avoids abelian cubes
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Our experimental results
- We find 5% of morphic words with additive and non-abelian squares
- All morphisms are similar to $\varphi_0$
Apply it to other morphisms

The corresponding incidence matrix:

\[ \text{Mat}(\varphi_0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]
Apply it to other morphisms

\[ \varphi_0(0) = 03 \quad \varphi_0(1) = 43 \]
\[ \varphi_0(3) = 1 \quad \varphi_0(4) = 01 \]

The corresponding incidence matrix:

\[
\text{Mat}(\varphi_0) = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

\[ \varphi(6) = 60 \quad \varphi(2) = 10 \]
\[ \varphi(0) = 2 \quad \varphi(1) = 62 \]

\[ \mathbf{w} = \lim_{n \to \infty} \varphi^n(6) = 602106226010106026226226021060101060101 \cdots \]
Apply previous proof to other morphisms

**Lemma**
If a morphism is similar to \( \varphi_0 \), then it fits the informatic proof developed by Cassaigne et al. in 2014.

**Theorem (Jamet, L., Stoll)**
Let \( w \) be a fixed point of a morphism similar to \( \varphi_0 \). The following propositions are decidable:

- \( w \) avoids additive cubes
- In \( w \), all additive squares are abelian squares
Why do we choose morphic words?

\[
\begin{align*}
\varphi(0) &= 2 \\
\varphi(1) &= 62 \\
\varphi(2) &= 10 \\
\varphi(6) &= 60
\end{align*}
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- \( \varphi^1(6) = 60 \)
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- \( \varphi^4(6) = 60210622 \)
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\[
\begin{itemize}
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  \item \varphi^3(6) = 60210
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### Why do we choose morphic words?

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\[w = 602106226010106026226226021060101060101\]

<table>
<thead>
<tr>
<th>(w[p])</th>
<th>6</th>
<th>0</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>6</th>
<th>2</th>
<th>2</th>
<th>6</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>(\text{par}(p))</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>
Parikh vector

The Parikh vector $\psi(x)$ of a word $x$ is:

$$\psi(x) = \begin{pmatrix} |x|_0 \\ |x|_1 \\ |x|_2 \\ |x|_6 \end{pmatrix}$$

example: $\psi(60210) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$
Where the alphabet matters

**Parikh vector**

The Parikh vector $\psi(x)$ of a word $x$ is:

$$\psi(x) = \begin{pmatrix} |x|_0 \\ |x|_1 \\ |x|_2 \\ |x|_6 \end{pmatrix}, \text{ example: } \psi(60210) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

If $b$ and $c$ are two blocks with same length and same sum then the vector $v = \psi(b) - \psi(c)$ belongs to the lattice

$$\mathcal{L} := \{v \in \mathbb{Z}^4 : (1, 1, 1, 1) \cdot v = 0 \text{ et } (0, 1, 2, 6) \cdot v = 0\}.$$ 

which depends on the chosen alphabet.
Linear algebra

Let $p$ be a position, we define

$$\sigma(p) = \psi(w[0, p]) = \begin{pmatrix} |w[0, p]|_0 \\ |w[0, p]|_1 \\ |w[0, p]|_2 \\ |w[0, p]|_6 \end{pmatrix}, \text{ example: } \sigma(9) = \psi(602106226) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$
Linear algebra

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Lemma (J. Cassaigne et al., 2014)

If $q$ is a child of $p$ and $a$ the proper prefix linked to $q$ (via the bijection), we get

$$\sigma(q) = M\sigma(p) + \psi(a)$$

Example :

$$\sigma(9) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \sigma(5) = \psi(60210) = \psi(6)$$
Corollary (J. Cassaigne et al., 2014)

If \( \{p_i\}_{i=0}^{\infty} \) is the ancestral sequence of a position \( p \) and denoting \( a_i \) the proper prefix used to link \( p_i \) to \( p_{i+1} \), we get: \( \sigma(p_0) = \sum_{i=0}^{\infty} M^i \psi(a_i) \).
So, how does it work?

- Using parents and graphs, we get bounds for $v = \psi(b) - \psi(c)$
- Using the lattice, we get other bounds for $v$
- $v$ lies in a ball of fixed radius
- This ball allow us to consider a finite subgraph
- We detect additive cubes by computing
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Using exactly the same arguments but considering two consecutive blocks rather than three, it is possible to detect additive squares
Statistics

Morphisms of size 2

- 32068 morphisms avoiding additive cubes, over 4-letters alphabets included in \{0, 1, \ldots, 25\}
- Less than 5% with a fixed point containing additive non-abelian squares
- 23 morphisms avoiding additive cubes over \{0, 1, 5, 25\}
- 2 morphisms avoiding additive cubes over \{0, 2, 5, 11\}
- At least one morphism for each alphabet included in \{0, 1, \ldots, 25\} except \{0, 1, 2, 3\} and \{0, 1, 2, 4\}.
- All morphisms avoiding additive cubes are similar to \varphi_0
Statistics

Morphisms of size 3

- 132 morphisms over 4-letters alphabets \{0, 1, 2, c\} (4 \leq c \leq 9) avoiding additive cubes
- Not all similar to \(\varphi_0\) : there is an other class
- 9 morphisms avoiding additive cubes over the alphabet \{0, 1, 2, 4\}, 5 are similar to \(\varphi_0\)
Proposition (Jamet, L., Stoll)

The following morphisms avoid additive cubes:

\[ \varphi_2 : \begin{cases} 
0 & \mapsto 21 \\
1 & \mapsto 011 \\
2 & \mapsto 214 \\
4 & \mapsto 244 
\end{cases} \quad \text{and} \quad \varphi_3 : \begin{cases} 
0 & \mapsto 4 \\
1 & \mapsto 12 \\
2 & \mapsto 0 \\
4 & \mapsto 100 
\end{cases} \]

Where \( \varphi_3 \) is similar to \( \varphi_0 \)
Proposition (Jamet, L., Stoll)

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Question

If \( \varphi \) is a morphism avoiding additive cubes, do there exist integers \( k \) and \( n \) such that:

\[ \varphi^k \simeq \varphi_0^n \]
Thank you for your attention