Rigidity and substitutive dendric words

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Joint work with
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Contain of:

V. Berthé, F. Dolce, F. Durand, J. Leroy, D. Perrin
Rigidity and Substitutive Dendric Words
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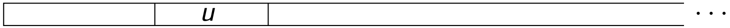
Dendric words = Tree words/sets in

V. Berthé, *et al,*
Acyclic, connected and tree sets, *Monatsh. Math., 2015*
Maximal bifix decoding, *Discrete Math., 2015*
The finite index basis property, *J. Pure Appl. Algebra, 2015*
⋮

Extension graphs

Take $\mathbf{w} \in A^{\mathbb{N}}$ and $u \in \mathrm{Fac}(\mathbf{w})$.

$\mathbf{w} =$ 

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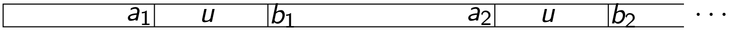
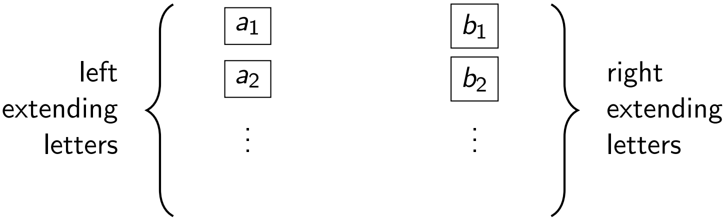
$\mathbf{w} =$ 

The extention graph of u in $\mathbf{w}$ is the undirected graph $E_{\mathbf{w}}(u) = (V, E)$, where

- $V = \{a \in A \mid au \in \mathrm{Fac}(\mathbf{w})\} \sqcup \{b \in A \mid ub \in \mathrm{Fac}(\mathbf{w})\}$;



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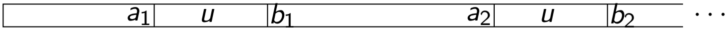
$\mathbf{w} =$ 

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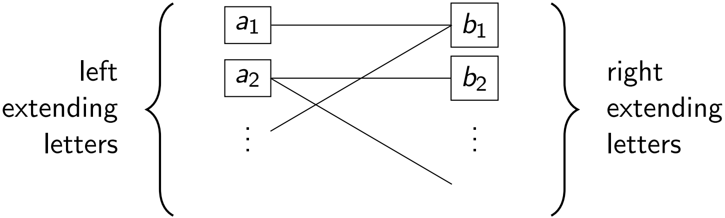
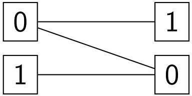


Extension graphs of dendric words

Example (Fibonacci)

$$\mathbf{f} = 010\boxed{010}10\boxed{010}01\boxed{010}010100100101001001010\cdots$$

$E_{\mathbf{f}}(010) =$



$10101 \notin \mathrm{Fac}(\mathbf{f})$

Extension graphs of dendric words

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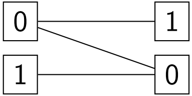


$10101 \notin \mathrm{Fac}(\mathbf{f})$

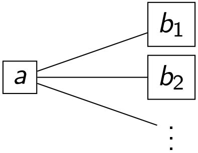
Definition

$\mathbf{w} \in A^{\mathbb{N}}$ is dendric if $E_{\mathbf{w}}(u)$ is a tree for all $u \in \mathrm{Fac}(\mathbf{w})$

Remark

$\mathbf{w}$ is dendric if $E_{\mathbf{w}}(u)$ is a tree for all bispecial factors u

Non-left special factors



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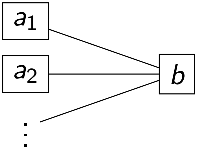
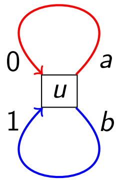


Various families of dendric words

- Sturmian words are dendric

$G_n(\mathbf{w})$



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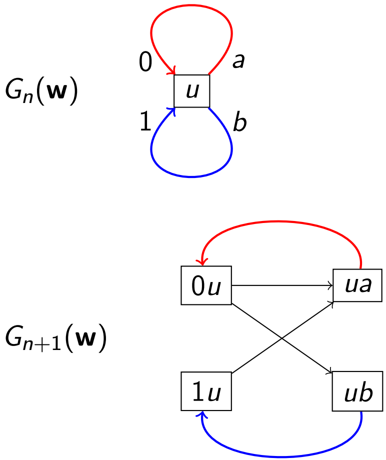

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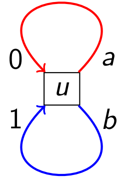
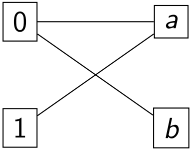


$E_{\mathbf{w}}(u)$



Various families of dendric words

- Sturmian words are dendric
- Arnoux-Rauzy words are dendric

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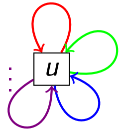
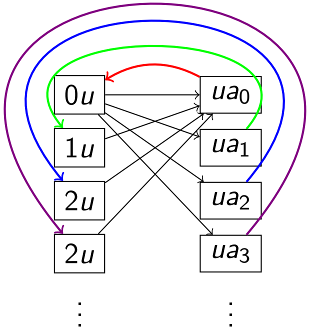


$G_{n+1}(\mathbf{w})$



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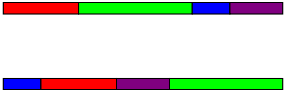
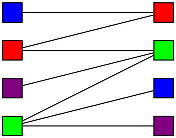
- Arnoux-Rauzy words are dendric
- Interval exchange words are dendric

4-IET

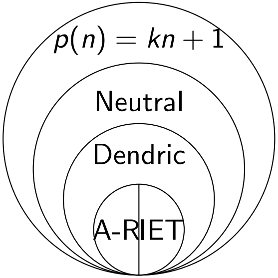


$E_{\mathbf{w}}(\varepsilon)$



Various families of dendric words

- Sturmian words are dendric
- Arnoux-Rauzy words are dendric
- Interval exchange words are dendric
- There is more



$$\begin{cases} a \mapsto ac \\ b \mapsto bac \\ c \mapsto cbac \end{cases}$$

Known properties of dendric words

- Recurrence = uniform recurrence

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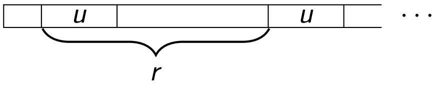
- Return words are bases of the free group



$$\mathcal{R}_{\mathrm{Fibo}}(10) = \{10, 100\}$$

$$0 = (10)^{-1} \cdot 100$$
$$1 = 10 \cdot 0^{-1}$$

Known properties of dendric words

- Recurrence = uniform recurrence
- Return words are bases of the free group
- Tame S-adic representations

S-adic representation:
$\mathbf{w} = \lim_{n\to+\infty} \sigma_0\sigma_1 \cdots \sigma_n(a)$

Tame: substitutions are

$$\alpha_{a,b} : \begin{cases} a \mapsto ab \\ c \mapsto c \neq a \end{cases}$$

$$\tilde{\alpha}_{a,b} : \begin{cases} a \mapsto ba \\ c \mapsto c \neq a \end{cases}$$

$$E_{a,b} : \begin{cases} a \mapsto b \\ b \mapsto a \\ c \mapsto c \neq a, b \end{cases}$$

Known properties of dendric words

- Recurrence = uniform recurrence
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- Tame S-adic representations
- Closed under maximal bifix decoding

Code:
$C \subset A^*$ s.t.
$u_1 \cdots u_k = v_1 \cdots v_l$
$\Downarrow$
$k = l$ and $u_i = v_i$ for all i

Bifix code:

code such that no word is prefix or suffix of another.

Decoding:
C = code, B = alphabet
$f : B \to C$ bijection
$\mathbf{w} \in C^{\mathbb{N}} \mapsto f^{-1}(\mathbf{w}) \in B^{\mathbb{N}}$

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$\vdots$

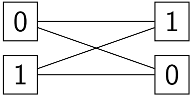
Questions: which words can be dendric ? In particular, what happen for substitutive words?

Not any substitutive word is dendric

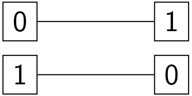
Example (Thue-Morse)

$\mathbf{t} = 0110100110010110100101100110100110010110 \cdots$

$E_{\mathbf{t}}(\varepsilon)$



$E_{\mathbf{t}}(010)$



Thue-Morse word is not dendric for good reasons

Theorem (Berthé, Dolce, Durand, L., Perrin)

If σ is primitive and constant length, then $\sigma^{\omega}(a)$ is not dendric.

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Theorem (Berthé, Dolce, Durand, L., Perrin)

If σ is primitive and constant length, then $\sigma^\omega(a)$ is not dendric.

A system (X, S) is totally minimal if (X, S^n) is minimal for all n.

Lemma

A minimal system (X, S) is not *totally minimal iff X has a cyclic partition:*

$$X = \bigcup_{i=0}^{n-1} X_i \quad \textit{and} \quad S(X_i) = X_{i+1 \bmod n}.$$

Sketch of the proof of Theorem.

- Minimal dendric shifts are totally minimal
- Take $X_0 = \bigcup_{a \in A} \sigma([a])$
  $X_0, S(X_0), \ldots, S^{|\sigma|-1}(X_0)$ is a cyclic partition of (X_σ, S)

□

But one can decide it

Theorem (Dolce, Kyriakoglou, L.)

If σ is a primitive substitution, one can decide whether $\sigma^\omega(a)$ is dendric or not.

Towards a "canonical" substitution

Definition

The stabilizer of $\mathbf{w} \in A^{\mathbb{N}}$ is the monoid of substitutions that generate $\mathbf{w}$, i.e.

$$\mathrm{Stab}(\mathbf{w}) = \{\tau : A^* \to A^* \mid \exists a \in A : \tau^\omega(a) = \mathbf{w}\} \cup \{\mathrm{id}\}$$

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$\mathbf{w}$ is rigid if $\mathrm{Stab}(\mathbf{w})$ is cyclic, i.e. there exists $\tau : A^* \to A^*$ s.t.

$$\mathrm{Stab}(\mathbf{w}) = \{\tau^n \mid n \in \mathbb{N}\}.$$

Remark

Not the same notion as rigidity in dynamics.

Question: are dendric words rigid?

Particular cases are known

Theorem (Séébold 1998, Richomme and Séébold 2012, Rao and Wen 2010)

Sturmian words are rigid.

Theorem (Krieger 2008)

Characteristic Arnoux-Rauzy words are rigid.

Proofs are not directly generalizable for dendric words.

Partial result for the general case

Theorem (Berthé, Dolce, Durand, L., Perrin)

Let $\mathbf{w}$ *be a dendric word.*

1. *If* $\sigma, \tau \in \mathrm{Stab}(\mathbf{w})$ *are primitive, then*
$$\sigma^m = \tau^n \quad \text{for some } m, n > 0.$$
2. *If* $\mathbf{w}$ *is recurrent and substitutive, then there exists* θ *primitive and tame s.t. for any primitive* $\sigma \in \mathrm{Stab}(\mathbf{w})$, *there is a tame substitution* τ *s.t.*
$$\sigma^m = \tau\theta^n\tau^{-1} \quad \text{for some } m, n > 0.$$

The proof uses return words and S-adic representations

Many questions remain open

- Rigidity is not solved. In particular, we have information only on the primitive substitution in the stabilizer. Is there something else?
- Could we describe dendric substitutions, i.e., substitutions that preserve dendricity?
- Are standard tame substitutions of some interest (like epistandard substitutions)?

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Thank you