Computing the entropy of multidimensional subshifts of finite type

Silvère Gangloff

LIP, ENS Lyon

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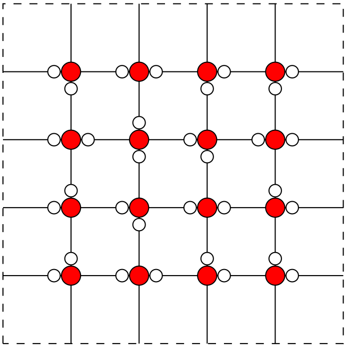
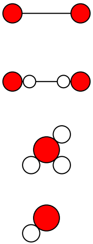
Bidimensional ice stable states [Pauling-Lieb]:



Forbidden:



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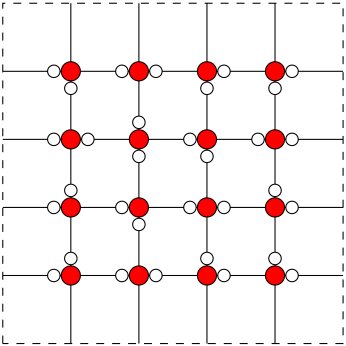

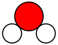

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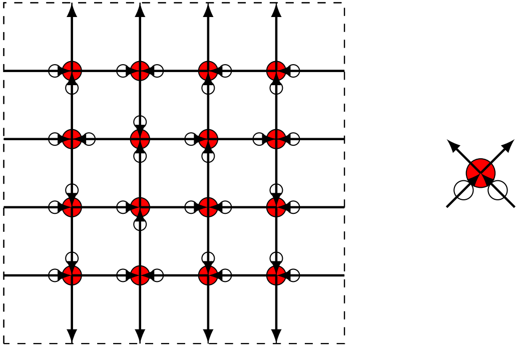

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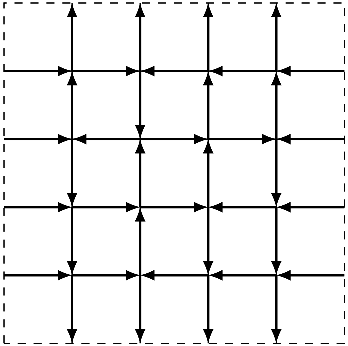

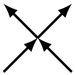

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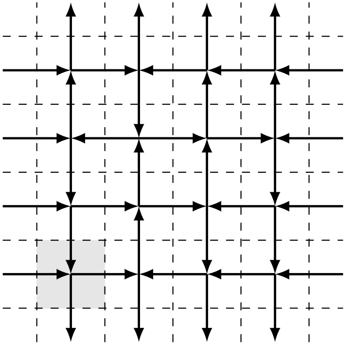

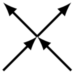

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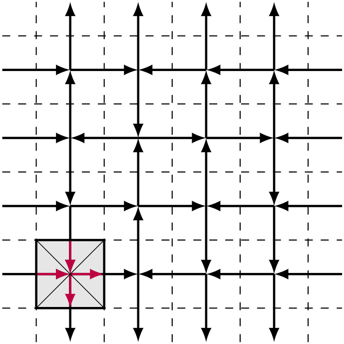

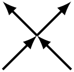

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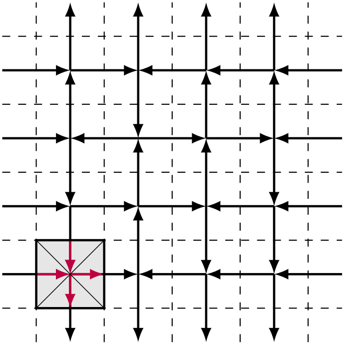
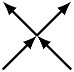
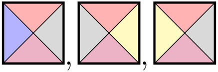
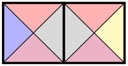
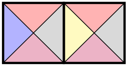




H. Wang Tiles (1960'):



possible: 

not possible: 

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

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0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	0	0	1	0
1	0	0	0	0

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oops

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Computing physical quantities related to the model?

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Entropy: "quantity of possible states".

$N_n(X)$: number of observable n size squares.

Free tiles

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1	0

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Hard core

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Hard core

1	0
0	1

$$N_2(X) = 7$$
$$N_n(X) = 2^{n^2(h(X)+o(1))}$$

Entropy of a SFT X:

$$h(X) = \inf_{n \geq 1} \frac{\log_2(N_n(X))}{n^2}$$

Free tiles	**Hard core**	**Ice [Lieb 67]**
$h = 1$	$h \geq 1/2$	$h = (4/3)^{3/2}$?

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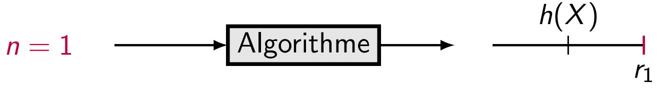

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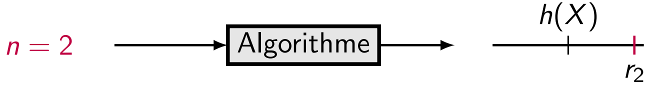

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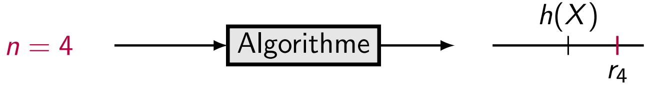

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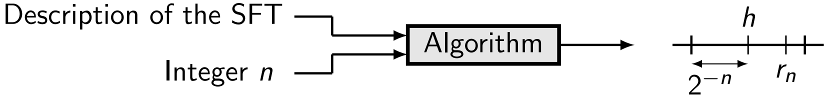
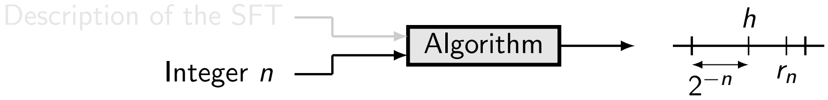


Implies computability:



Théorème (Hochman, Meyerovitch 2010)

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$$\sum_{k \in \mathcal{H}} 2^{-k},$$

$\mathcal{H}$: programs which never stop on $n = 0$.

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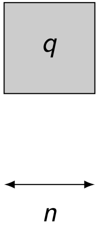

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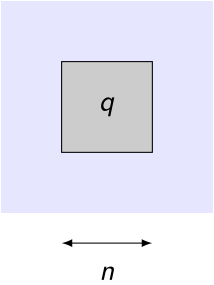

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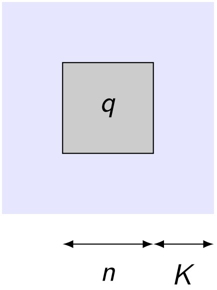

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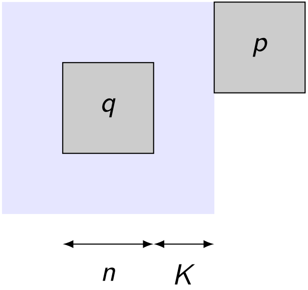

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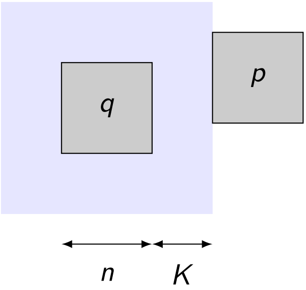

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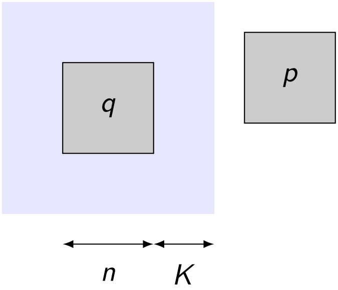

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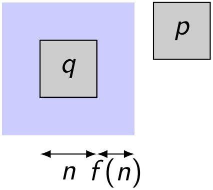

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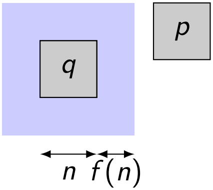

Théorème (G., Sablik)

1. *The possible values of entropy for* **linear** *block gluing ($f(n) = O(n)$,* **ex**: *ice) are the non-negative numbers semi-computable from above.*
2. *If $f(n) = o(\log(n))$, entropy is computable.*

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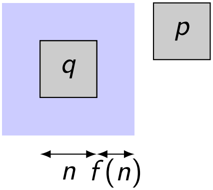

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Answer to **problem 9.1** of Hochman and Meyerovitch.

Characterisation of a threshold for f?

For **decidable subshifts** (not including all SFT), meaning that $n \mapsto N_n(X)$ is computable:

Théorème (G.,Hellouin)

Let $f : \mathbb{N} \to \mathbb{N}$ be non-decreasing.

1. $\sum_n \frac{f(n)}{n^2} < +\infty$ *with computable speed: the entropy is computable.*
2. $\sum_n \frac{f(n)}{n^2} = +\infty$*: possible values: semi-computable numbers from above.*

Questions left open:

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Existence of a threshold for multidimensional SFT?

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Characterisation of the possible values under the threshold ?

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General aim: understand the conditions under which we can compute the entropy of physical pertinent models.