k-Spectra of c-Balanced Words

k-Spectra

Pamela Fleischmann, Joel D. Day

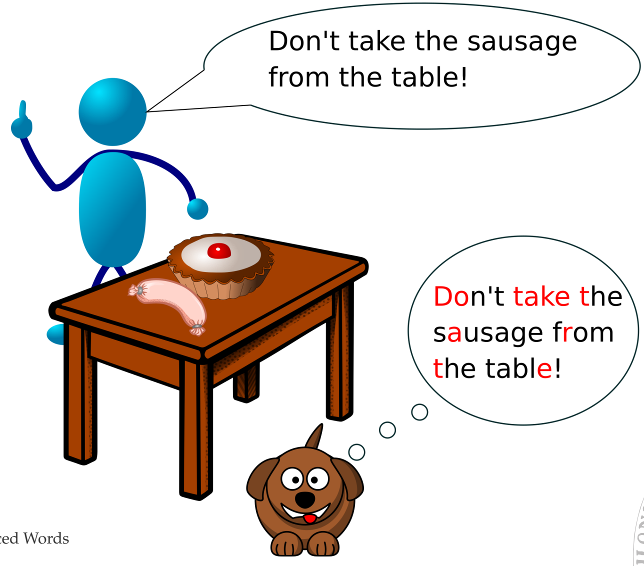
{fpa,jda}@informatik.uni-kiel.de

September, 13th

Kiel University

Dependable Systems Group

Information Loss



Scattered Factors

informal: deleting arbitrary letters from a word (preserving the order) results in a scattered factor of this word

Scattered Factors

informal: deleting arbitrary letters from a word (preserving the order) results in a scattered factor of this word

Example

b a n a n a

Scattered Factors

informal: deleting arbitrary letters from a word (preserving the order) results in a scattered factor of this word

Example

a a a

Scattered Factors

informal: deleting arbitrary letters from a word (preserving the order) results in a scattered factor of this word

Example

b a n a n a

Scattered Factors

informal: deleting arbitrary letters from a word (preserving the order) results in a scattered factor of this word

Example

b a n a

Scattered Factors

informal: deleting arbitrary letters from a word (preserving the order) results in a scattered factor of this word

Definition (Scattered Factor, (Scattered) Subword)

$v = v_1 \dots v_n \in \Sigma^*$ scattered factor of w iff

$$\exists u_0 \dots u_n \in \Sigma^* : \; w = u_0 v_1 u_1 \dots u_{n-1} v_n u_n .$$

k-Spectra

Definition

- set of all scattered factors of w is the spectrum $\mathrm{ScatFact}(w)$

k-Spectra

Definition

- set of all scattered factors of w is the spectrum $\mathrm{ScatFact}(w)$
- set of all scattered factors of w of length k is the k-spectrum $\mathrm{ScatFact}_k(w)$

k-Spectra

Definition

- set of all scattered factors of w is the spectrum $\mathrm{ScatFact}(w)$
- set of all scattered factors of w of length k is the k-spectrum $\mathrm{ScatFact}_k(w)$

Example: abba

{abba}	4-spectrum
{aba, bba, abb}	3-spectrum
{aa, ab, bb, ba}	2-spectrum
{a, b}	1-spectrum

Definition

- set of all scattered factors of w is the spectrum $\mathrm{ScatFact}(w)$
- set of all scattered factors of w of length k is the k-spectrum $\mathrm{ScatFact}_k(w)$

We are not considering multisets.

Open Problems

Problem

Given $S \subseteq \Sigma^*$ decide whether S is the spectrum of some word w.

Open Problems

Problem

Given $S \subseteq \Sigma^*$ decide whether S is the spectrum of some word w.

Problem

Given a k-spectrum decide whether it is independent, *e.g.* $\{\mathtt{ab}, \mathtt{ba}, \mathtt{aa}\}$ *is not independent since* aa *can be deduced from* ab *and* ba.

Open Problems

Problem

Given $S \subseteq \Sigma^*$ *decide whether* S *is the spectrum of some word* w.

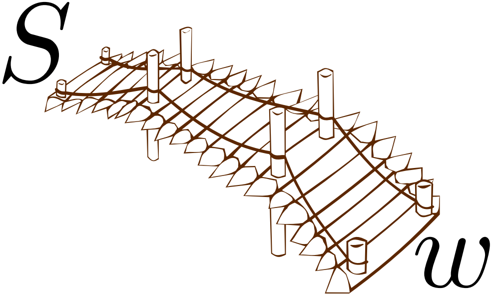
Problem

Given a k*-spectrum decide whether it is* independent, *e.g.* $\{\mathsf{ab}, \mathsf{ba}, \mathsf{aa}\}$ *is not independent since* aa *can be deduced from* ab *and* ba.

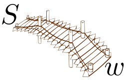
Problem

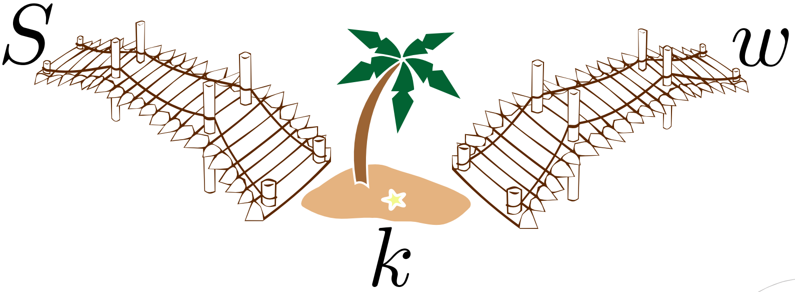
Determine the index of the equivalence relation that relates word with the same spectrum.

Middle Step Between S and w



Middle Step Between S and w





Reformulated Problem

Problem

Decide for a given $n \in \mathbb{N}$ *whether there exists* $w \in \Sigma^*$ *and* $k \in \mathbb{N}$ *with* $|\mathrm{ScatFact}_k(w)| = n$.

Reformulated Problem

Problem

Decide for a given $n \in \mathbb{N}$ *whether there exists* $w \in \Sigma^*$ *and* $k \in \mathbb{N}$ *with* $|\mathrm{ScatFact}_k(w)| = n$.

or more restricted:

Problem

Decide for given $n, k \in \mathbb{N}$ *whether there exists* $w \in \Sigma^*$ *with* $|\mathrm{ScatFact}_k(w)| = n$.

Reformulated Problem

Problem

Decide for a given $n \in \mathbb{N}$ *whether there exists* $w \in \Sigma^*$ *and* $k \in \mathbb{N}$ *with* $|\mathrm{ScatFact}_k(w)| = n$.

or more restricted:

Problem

Decide for given $n, k \in \mathbb{N}$ *whether there exists* $w \in \Sigma^*$ *with* $|\mathrm{ScatFact}_k(w)| = n$.

To start with we only consider a binary alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

Examples

- $n = 3, k = 2$: $w =$ aabb

Examples

- $n = 3, k = 2$: $w = \mathtt{aabb}$
- $n = k + 2, k \in \mathbb{N}_{>2}, |w|_a = |w|_b$ does not have a solution

Examples

- $n = 3, k = 2$: $w = \mathsf{aabb}$
- $n = k + 2, k \in \mathbb{N}_{>2}, |w|_a = |w|_b$ does not have a solution
- $n = 2^k, k \in \mathbb{N}$: $w = (\mathsf{ab})^k$

Examples

- $n = 3, k = 2$: $w = \mathsf{aabb}$
- $n = k + 2, k \in \mathbb{N}_{>2}, |w|_a = |w|_b$ does not have a solution
- $n = 2^k, k \in \mathbb{N}$: $w = (\mathsf{ab})^k$
- n square number at least 4: $k := 2(\sqrt{n} - 1)$, $w = \mathsf{a}^{\frac{k}{2}}\mathsf{b}^k\mathsf{a}^{\frac{k}{2}}$

c-balanced words

Definition

Binary word $w \in \{\mathsf{a}, \mathsf{b}\}^*$ *c-balanced* for a $c \in \mathbb{N}_0$ iff

$$\big|\, |w|_{\mathsf{a}} - |w|_{\mathsf{b}} \,\big| = c.$$

c-balanced words

Definition

Binary word $w \in \{\mathsf{a}, \mathsf{b}\}^*$ c-balanced for a $c \in \mathbb{N}_0$ iff

$$\big|\, |w|_{\mathsf{a}} - |w|_{\mathsf{b}} \,\big| = c.$$

- $c = 1$: w balanced

c-balanced words

Definition

Binary word $w \in \{\mathsf{a}, \mathsf{b}\}^*$ *c-balanced* for a $c \in \mathbb{N}_0$ iff

$$\big|\, |w|_{\mathsf{a}} - |w|_{\mathsf{b}} \,\big| = c.$$

- $c = 1$: w *balanced*
- $c = 0$: w *strictly balanced*

c-balanced words

Definition

Binary word $w \in \{\mathsf{a}, \mathsf{b}\}^*$ c-balanced for a $c \in \mathbb{N}_0$ iff

$$\big|\, |w|_{\mathsf{a}} - |w|_{\mathsf{b}} \,\big| = c.$$

- $c = 1$: w balanced
- $c = 0$: w strictly balanced

Obviously for every $w \in \{\mathsf{a}, \mathsf{b}\}$ exists $c \in \mathbb{N}_0$ such that w is c-balanced.

Pecularities of Restriction to Cardinalities

Example: 3-spectrum

abbab	baaba	babba	abaab
	aaa		aaa
aab	aab		aab
aba	aba	aba	aba
abb		abb	abb
	baa	baa	baa
bab	bab	bab	bab
bba	bba	bba	
bbb		bbb	
6	6	6	6

Pecularities of Restriction to Cardinalities

Example: 3-spectrum

abbab	baaba	babba	abaab
	aaa		aaa
aab	aab		aab
aba	aba	aba	aba
abb		abb	abb
	baa	baa	baa
bab	bab	bab	bab
bba	bba	bba	
bbb		bbb	
6	6	6	6

	renaming	reverse	both
abbab	baaba	babba	abaab
aab	bba	baa	abb
aba	bab	aba	bab
abb	baa	bba	aab
bab	aba	bab	aba
bba	aab	abb	baa
bbb	aaa	bbb	aaa
6	6	6	6

Renaming and Reversing

Definition

- $\bar{\cdot} : \Sigma \to \Sigma$ with $\bar{\mathsf{a}} = \mathsf{b}$ and $\bar{\mathsf{b}} = \mathsf{a}$ renaming morphism

Renaming and Reversing

Definition

- $\overline{\cdot} : \Sigma \to \Sigma$ with $\overline{\mathsf{a}} = \mathsf{b}$ and $\overline{\mathsf{b}} = \mathsf{a}$ renaming morphism
- $\cdot^R : \Sigma^* \to \Sigma^*$ with $w^R = w[|w|] \ldots w[1]$ with the i^{th} letter $w[i]$ of w

Renaming and Reversing

Definition

- $\bar{\cdot} : \Sigma \longrightarrow \Sigma$ with $\bar{\mathsf{a}} = \mathsf{b}$ and $\bar{\mathsf{b}} = \mathsf{a}$ renaming morphism
- $\cdot^R : \Sigma^* \longrightarrow \Sigma^*$ with $w^R = w[|w|] \ldots w[1]$ with the i^{th} letter $w[i]$ of w

Lemma

- $\text{ScatFact}(\overline{w}) = \{\overline{u} \mid u \in \text{ScatFact}(w)\}$

Renaming and Reversing

Definition

- $\bar{\cdot} : \Sigma \longrightarrow \Sigma$ with $\bar{\mathsf{a}} = \mathsf{b}$ and $\bar{\mathsf{b}} = \mathsf{a}$ renaming morphism
- $\cdot^R : \Sigma^* \longrightarrow \Sigma^*$ with $w^R = w[|w|] \dots w[1]$ with the i^{th} letter $w[i]$ of w

Lemma

- $\text{ScatFact}(\overline{w}) = \{\overline{u} \mid u \in \text{ScatFact}(w)\}$
- $\text{ScatFact}(w^R) = \{u^R \mid u \in \text{ScatFact}(w)\}$

Pecularities of Restriction to Cardinalities

Corollary

The cardinalities of the spectra (and k-spectra) of w, w^R, and $\overline{w}$ are the same:

$$|\mathrm{ScatFact}_k(w)| = |\mathrm{ScatFact}_k(w^R)| = |\mathrm{ScatFact}_k(\overline{w})|.$$

Pecularities of Restriction to Cardinalities

Corollary

The cardinalities of the spectra (and k-spectra) of w, w^R, and $\overline{w}$ are the same:

$$|\,\mathrm{ScatFact}_k(w)| = |\,\mathrm{ScatFact}_k(w^R)| = |\,\mathrm{ScatFact}_k(\overline{w})|.$$

$\mathtt{a} < \mathtt{b}$ assumed: only consider the lexicographically smallest element in such a equivalence class

Solving the first problem

Theorem

For all $n \in \mathbb{N}$ the k-spectrum of $w = \mathtt{a}^k\mathtt{b}^k$ for $k = n-1$ has n elements, i.e. $|\mathrm{ScatFact}_{n-1}(\mathtt{a}^{n-1}\mathtt{b}^{n-1})| = n$.

Solving the first problem

Theorem

For all $n \in \mathbb{N}$ the k-spectrum of $w = \mathtt{a}^k\mathtt{b}^k$ for $k = n-1$ has n elements, i.e. $|\mathrm{ScatFact}_{n-1}(\mathtt{a}^{n-1}\mathtt{b}^{n-1})| = n$.

Proof:

- all $\mathtt{a}^r\mathtt{b}^s$ for $r + s = n - 1$ are the scattered factors of length $n - 1$

Solving the first problem

Theorem

For all $n \in \mathbb{N}$ the k-spectrum of $w = \mathsf{a}^k\mathsf{b}^k$ for $k = n - 1$ has n elements, i.e. $|\mathrm{ScatFact}_{n-1}(\mathsf{a}^{n-1}\mathsf{b}^{n-1})| = n$.

Proof:

- all $\mathsf{a}^r\mathsf{b}^s$ for $r + s = n - 1$ are the scattered factors of length $n - 1$
- n possibilities □

Solving the first problem

Theorem

For all $n \in \mathbb{N}$ *the* k*-spectrum of* $w = \mathsf{a}^k\mathsf{b}^k$ *for* $k = n-1$ *has* n *elements, i.e.* $|\mathrm{ScatFact}_{n-1}(\mathsf{a}^{n-1}\mathsf{b}^{n-1})| = n$.

Corollary

$S_n = \{\mathsf{a}^r\mathsf{b}^s \,|\, r+s=n \in \mathbb{N}\}$ *is a scattered factor set for all* $n \in \mathbb{N}$.

Partly Solving the Second Problem

Theorem

Given $k, n \in \mathbb{N}$ *with* $n - 1 \leq k$ *set* $c = k - n + 1$ *and consider* $w = \mathsf{a}^k\mathsf{b}^{k-c}$. *Then for all* $i \in [c]_0$ *the* $(k - i)$*-spectrum of* w *has cardinality* n.

Partly Solving the Second Problem

Theorem

Given $k, n \in \mathbb{N}$ with $n - 1 \leq k$ set $c = k - n + 1$ and consider $w = \mathsf{a}^k\mathsf{b}^{k-c}$. Then for all $i \in [c]_0$ the $(k-i)$-spectrum of w has cardinality n.

Proof:

- $i = 0$: $\mathsf{a}^r\mathsf{b}^s$ with $r + s = k \rightsquigarrow k - c + 1 = n$ possibilities
- $i \neq 0$: all the scattered factor are just *shortened* for the $(k-i)$-spectra

Are we happy now?

- Given $n \in \mathbb{N}$ for each c we have $c+1$ different sets being a spectrum of cardinality n.

Are we happy now?

- Given $n \in \mathbb{N}$ for each c we have $c+1$ different sets being a spectrum of cardinality n.
- k-spectrum of $w = \mathsf{a}^2\mathsf{b}^4\mathsf{a}^2$ has cardinality 9

Are we happy now?

- Given $n \in \mathbb{N}$ for each c we have $c + 1$ different sets being a spectrum of cardinality n.
- k-spectrum of $w = \mathsf{a}^2\mathsf{b}^4\mathsf{a}^2$ has cardinality 9
- abba is a scattered factor of w and not in the aforementioned sets

Are we happy now?

- Given $n \in \mathbb{N}$ for each c we have $c + 1$ different sets being a spectrum of cardinality n.
- k-spectrum of $w = \mathsf{a}^2\mathsf{b}^4\mathsf{a}^2$ has cardinality 9
- abba is a scattered factor of w and not in the aforementioned sets
- which scattered factor sets have cardinality $n \in \mathbb{N}$

Are we happy now?

- Given $n \in \mathbb{N}$ for each c we have $c + 1$ different sets being a spectrum of cardinality n.
- k-spectrum of $w = \mathsf{a}^2\mathsf{b}^4\mathsf{a}^2$ has cardinality 9
- abba is a scattered factor of w and not in the aforementioned sets
- which scattered factor sets have cardinality $n \in \mathbb{N}$
- for a fixed $c \in \mathbb{N}$ and c-balanced words: which cardinalities are *reachable*

Are we happy now?

- Given $n \in \mathbb{N}$ for each c we have $c + 1$ different sets being a spectrum of cardinality n.
- k-spectrum of $w = \mathsf{a}^2\mathsf{b}^4\mathsf{a}^2$ has cardinality 9
- abba is a scattered factor of w and not in the aforementioned sets
- which scattered factor sets have cardinality $n \in \mathbb{N}$
- for a fixed $c \in \mathbb{N}$ and c-balanced words: which cardinalities are *reachable*

We were not happy! We would like to fully characterise for given c and word-length which cardinalities are reachable.

Minimal Cardinality

Lemma

For $w \in \Sigma^*$ and $k, c \in \mathbb{N}_0$ with $c \leq k$ we have

$$\forall i \in [c]_0 : \ |\mathrm{ScatFact}_{k-i}(w)| = k - c + 1 \quad \textit{iff} \quad w = \mathsf{a}^k \mathsf{b}^{k-c}.$$

Moreover $|\mathrm{ScatFact}_{k-i}(w)| \geq k - c + 1$ *for all* $i \in [c]_0$

Minimal Cardinality

Lemma

For $w \in \Sigma^*$ *and* $k, c \in \mathbb{N}_0$ *with* $c \leq k$ *we have*

$$\forall i \in [c]_0 : |\mathrm{ScatFact}_{k-i}(w)| = k - c + 1 \quad \textit{iff} \quad w = \mathsf{a}^k \mathsf{b}^{k-c}.$$

Moreover $|\mathrm{ScatFact}_{k-i}(w)| \geq k - c + 1$ *for all* $i \in [c]_0$

Proof idea for remaining part:

- suppose $w \neq \mathsf{a}^k \mathsf{b}^{k-c}$ (neither one of the symmetric cases)
- $\Rightarrow w = w_1 \mathsf{aba} w_2$
- induction on word-length

k-SPECTRA FOR STRICTLY BALANCED WORDS OF LENGTH $2k$

Properties of strictly balanced words

- same amount of as and bs

Properties of strictly balanced words

- same amount of as and bs
- always even length

Properties of strictly balanced words

- same amount of as and bs
- always even length
- the k-spectra has at most 2^k elements

Reaching the Maximal Cardinality

Lemma

The k-spectrum of a strictly balanced word $w \in \Sigma^*$ has cardinality 2^k iff $w \in \{\mathtt{ab}, \mathtt{ba}\}^k$, i.e.

$$|\mathrm{ScatFact}_k(w)| = 2^k \Leftrightarrow w \in \{\mathtt{ab}, \mathtt{ba}\}^k.$$

Reaching the Maximal Cardinality

Lemma

The k-spectrum of a strictly balanced word $w \in \Sigma^*$ has cardinality 2^k iff $w \in \{\mathsf{ab}, \mathsf{ba}\}^k$, i.e.

$$|\operatorname{ScatFact}_k(w)| = 2^k \Leftrightarrow w \in \{\mathsf{ab}, \mathsf{ba}\}^k.$$

Sketch of Proof:

Reaching the Maximal Cardinality

Lemma

The k-spectrum of a strictly balanced word $w \in \Sigma^*$ has cardinality 2^k iff $w \in \{\mathtt{ab}, \mathtt{ba}\}^k$, i.e.

$$|\,\mathrm{ScatFact}_k(w)| = 2^k \Leftrightarrow w \in \{\mathtt{ab}, \mathtt{ba}\}^k.$$

Sketch of Proof:

- "$\Rightarrow$" contraposition; if aa is a factor of w then one $\mathtt{b}^r\,\mathtt{a}^{k-r}$ is not a scattered factor

Reaching the Maximal Cardinality

Lemma

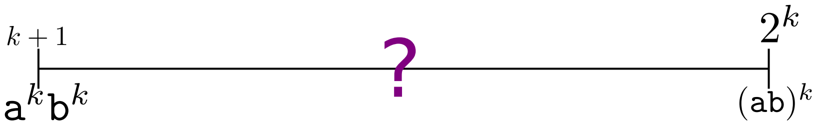
The k-spectrum of a strictly balanced word $w \in \Sigma^*$ has cardinality 2^k iff $w \in \{\mathsf{ab}, \mathsf{ba}\}^k$, i.e.

$$|\operatorname{ScatFact}_k(w)| = 2^k \Leftrightarrow w \in \{\mathsf{ab}, \mathsf{ba}\}^k.$$

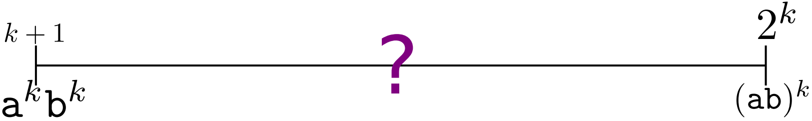
Sketch of Proof:

- "$\Rightarrow$" contraposition; if aa is a factor of w then one $\mathsf{b}^r\mathsf{a}^{k-r}$ is not a scattered factor
- "$\Leftarrow$" induction

Spectrum of k-spectra

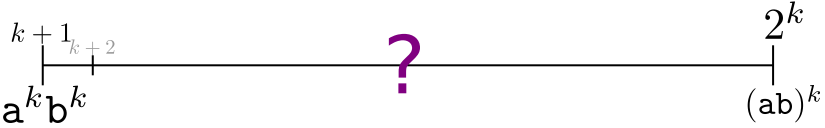


Spectrum of k-spectra



Proof for "$|\mathrm{ScatFact}_k(w)| = k + 1$ iff $w = \mathtt{a}^k\mathtt{b}^k$ gives also that $k + 2$ is not reachable!

Spectrum of k-spectra



Proof for "$|\mathrm{ScatFact}_k(w)| = k+1$ iff $w = \mathtt{a}^k\mathtt{b}^k$ gives also that $k+2$ is not reachable!

Jumbling a and b: $a^k b^k \longrightarrow (ab)^k$

if the as and bs are not nicely ordered we have at least one switch from b to a

Jumbling a and b: $a^k b^k \longrightarrow (ab)^k$

if the as and bs are not nicely ordered we have at least one switch from b to a

Lemma

The k-spectrum of a strictly balanced word $w \in \Sigma^*$ has cardinality $2k$ iff w is either $a^{k-1}bab^{k-1}$ or $a^{k-1}b^k a$, i.e.

$$|\mathrm{ScatFact}_k(w)| = 2k \Leftrightarrow w \in \{a^{k-1}bab^{k-1}, a^{k-1}b^k a\}.$$

Jumbling a and b: $a^k b^k \longrightarrow (ab)^k$

if the as and bs are not nicely ordered we have at least one switch from b to a

Lemma

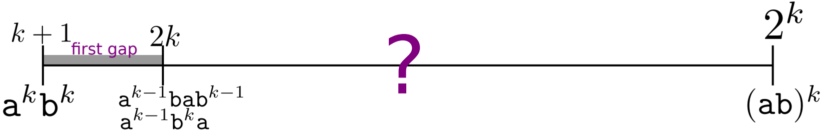
The k-spectrum of a strictly balanced word $w \in \Sigma^*$ has cardinality $2k$ iff w is either $a^{k-1}bab^{k-1}$ or $a^{k-1}b^k a$, i.e.

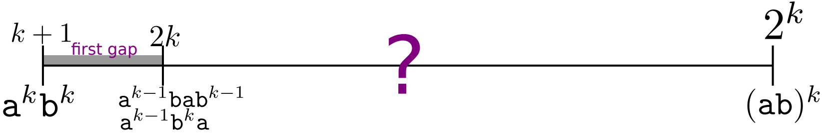
$$|\mathrm{ScatFact}_k(w)| = 2k \Leftrightarrow w \in \{a^{k-1}bab^{k-1}, a^{k-1}b^k a\}.$$

Our proof also shows

- If w is neither $a^k b^k$ nor $a^{k-1}bab^{k-1}$ nor $a^{k-1}b^k a$, then the cardinality is greater than $2k$

Spectrum of k-spectra

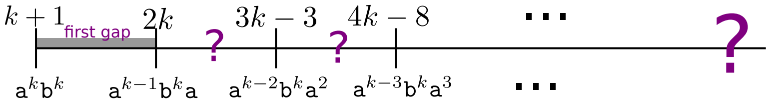




$\mathtt{a}^{k-1}\mathtt{b}^k\mathtt{a}$ generalisable to $\mathtt{a}^{k-i}\mathtt{b}^k\mathtt{a}^i$ for $i \in [\lfloor \frac{k}{2} \rfloor]$:

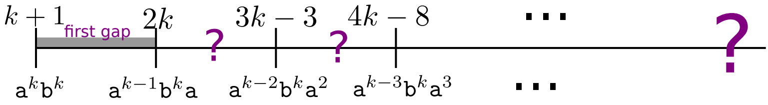
$$|\mathrm{ScatFact}_k(\mathtt{a}^{k-i}\mathtt{b}^k\mathtt{a}^i)| = k(i+1) - i^2 + 1$$

Spectrum of k-spectra



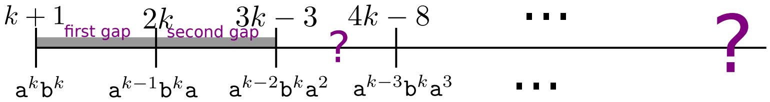
$a^{k-1}b^k a$ generalisable to $a^{k-i}b^k a^i$ for $i \in [\lfloor \frac{k}{2} \rfloor]$:

$$|\text{ScatFact}_k(a^{k-i}b^k a^i)| = k(i+1) - i^2 + 1$$



Promising news: the k-spectra of strictly balanced words cannot have cardinality $2k + i$ for $i \in [k - 4]$

Spectrum of k-spectra



Promising news: the k-spectra of strictly balanced words cannot have cardinality $2k + i$ for $i \in [k - 4]$

The third gap . . .

Raised hope: gap between $(i+1)k - i^2 + 1$ and $(i+2)k - (i+1)^2 + 1$

The third gap . . .

Raised hope: gap between $(i+1)k - i^2 + 1$ and $(i+2)k - (i+1)^2 + 1$

but (unfortunately)

The third gap . . .

Raised hope: gap between $(i+1)k - i^2 + 1$ and $(i+2)k - (i+1)^2 + 1$

but (unfortunately)

Lemma

The k-spectrum of $\mathsf{a}^{k-1}\mathsf{b}^2\mathsf{ab}^{k-2}$ *has exactly* $3k-2$ *elements.*

The third gap . . .

Raised hope: gap between $(i+1)k - i^2 + 1$ and $(i+2)k - (i+1)^2 + 1$

but (unfortunately)

Lemma

The k-spectrum of $\mathsf{a}^{k-1}\mathsf{b}^2\mathsf{a}\mathsf{b}^{k-2}$ *has exactly* $3k-2$ *elements.*

and this result is generalisable

The Thing in the "Gap"

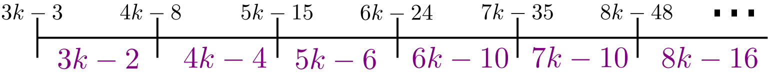
Lemma

For $k \geq 5$ and $i \in [k-1]$

- $|\mathrm{ScatFact}_k(\mathsf{a}^{k-2}\mathsf{b}^i\mathsf{a}\mathsf{b}^{k-i}\mathsf{a})| = k(2i+2) - 6i + 2$
- $|\mathrm{ScatFact}_k(\mathsf{a}^{k-2}\mathsf{b}^i\mathsf{a}^2\mathsf{b}^{k-i})| = k(2i+1) - 4i + 2$

Spectrum of k-spectra

$k \geq 38$



The other end of the spectrum

We saw already that the cardinality 2^k is reached iff $w = (\mathsf{ab})^k$.

The other end of the spectrum

We saw already that the cardinality 2^k is reached iff $w = (\mathsf{ab})^k$.

Lemma

The k-spectrum of w has cardinality $2^k - 1$ *iff* $w = (\mathsf{ab})^i \mathsf{a}^2 \mathsf{b}^2 (\mathsf{ab})^{k-i-2}$ *for some* $i \in [k-2]$.

The other end of the spectrum

We saw already that the cardinality 2^k is reached iff $w = (\mathsf{ab})^k$.

Lemma

The k-spectrum of w has cardinality $2^k - 1$ iff $w = (\mathsf{ab})^i \mathsf{a}^2\mathsf{b}^2(\mathsf{ab})^{k-i-2}$ for some $i \in [k-2]$.

Proof:

- "$\Leftarrow$" $\surd$
- "$\Rightarrow$" if there is a scattered factor not of the form $\mathsf{b}^{i+1}\mathsf{a}^{k-i-1}$ then less than $2^k - 1$ element are in the k-spectrum

Overview for strictly balanced words

k=	2	3	4	5	6	7	8	9
	3	4	5	6	7	8	9	10
	4	5	6	7	8	9	10	11
		6	7	8	9	10	11	12
		7	8	9	10	11	12	13
		8	9	10	11	12	13	14
			10	11	12	13	14	15
			11	12	13	14	15	16
			12	13	14	15	16	17
			13	14	15	16	17	18
			14	15	16	17	18	19
			15	16	17	18	19	20
			16	17	18	19	20	21
				18	19	20	21	22
				19	20	21	22	23
				20	21	22	23	24
				21	22	23	24	25
				22	23	24	25	26
				23	24	25	26	27
				24	25	26	27	28

9	8	7	6	5	4	3	2	=k
512	256	128	64	32	16	8	4	
511	255	127	63	31	15	7	3	
510	254	126	62	30	14	6		
509	253	125	61	29	13	5		
508	252	124	60	28	12	4		
507	251	123	59	27				
506	250	122	58	26				
505	249	121	⋮	25				
504	248	⋮	38					
503	⋮	54	37					
⋮	40	53						
82	39							
81								