# RIGIDITY FOR SOME DYNAMICAL SYSTEMS OF ARITHMETiC ORIGIN

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# THE QUESTION OF RIGIDITY

Rgidity = there exists a sequence $q_n \to \infty$ such that for any measurable set

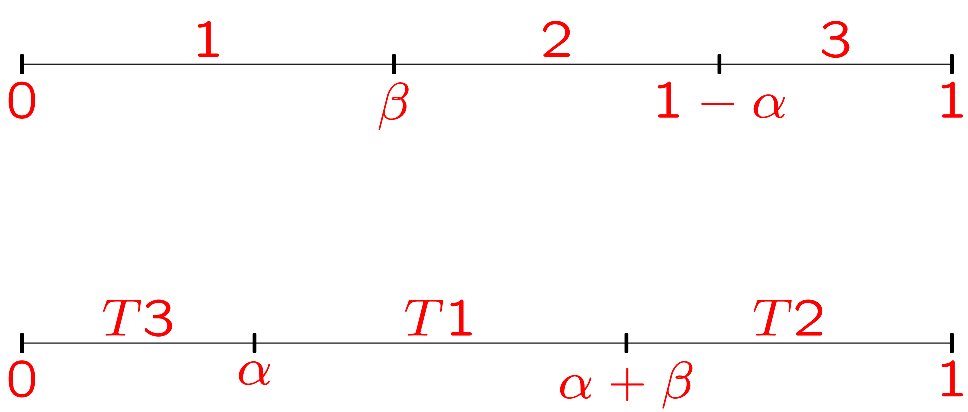
$\mu(T^{q_n} A \Delta A) \to 0.$

Veech (1982) : almost all interval exchanges are rigid.

Examples of non-rigid iet were known only for 3 intervals. Until Robertson (2017) and the square-tiled interval exchanges of F-H. (2016-17)

# A THREE-INTERVAL EXCHANGE

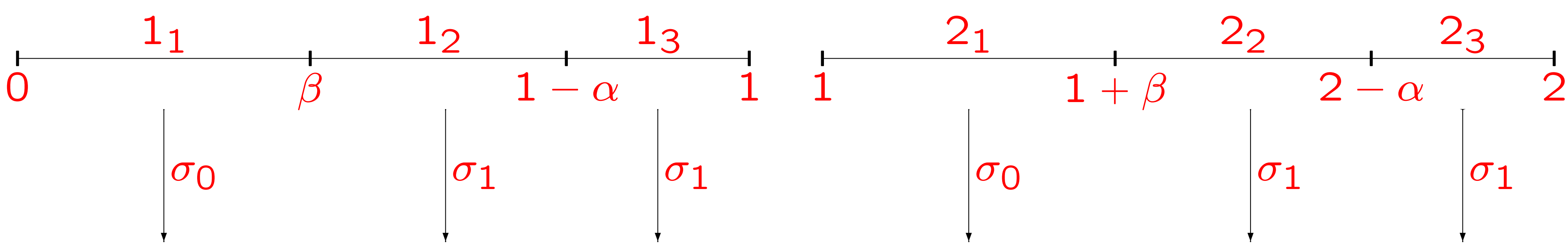
Take the rotation $Rx = x + \alpha$ modulo $1$ and mark a point $\beta$.



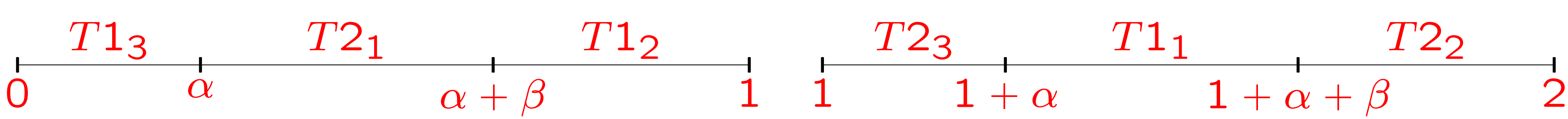
# THE EXAMPLE OF VEECH 1969

$T(x,s) = (Rx, \sigma_0 s)$ if $x$ is in the interval $[0,\beta[\times\{s\}$, assimilated with $[s-1, s-1+\beta[$,

$T(x,s) = (Rx, \sigma_1 s)$ if $x$ is in the interval $[\beta,1[\times\{s\}$, assimilated with $[s-1+\beta, s[$,

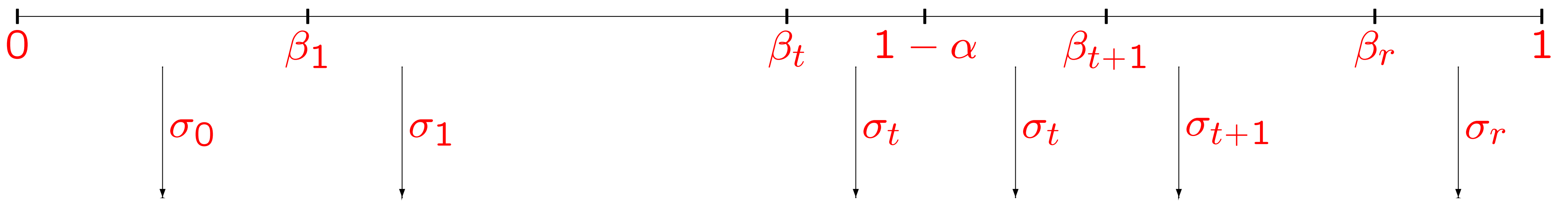


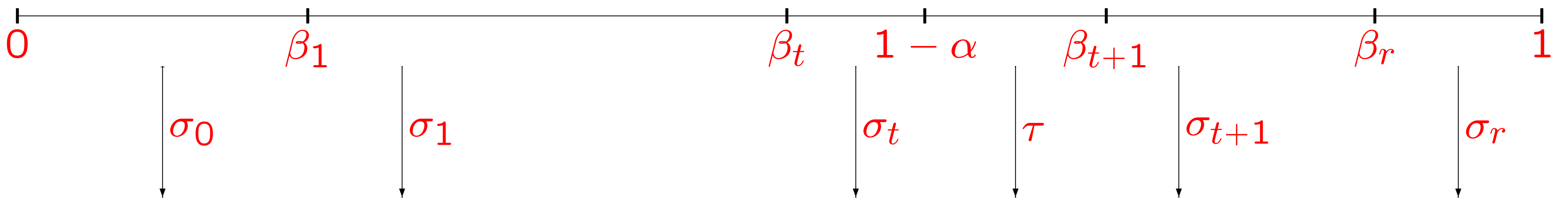

Thus the image intervals are



# GENERALIZED VEECH

We start from $R$, mark several points $\beta_i$, use permutations on $\{1, ...d\}$, take $d$ copies of the interval $[0, 1[$. Optionally, change permutations at $1 - \alpha$, like in square-tiled iet.



# GRAND UNIFICATION

We take $\alpha$ irrational, $0 = \beta_0 < \beta_1 < \ldots \beta_t < 1 - \alpha < \beta_{t+1} < \ldots \beta_r < \beta_{r+1} = 1$, $\sigma_0$, ..., $\sigma_r$, $\tau$, permutations of $\{1, \ldots d\}$.

$Rx = x + \alpha$ modulo $1$.

$T(x, s) = (x, \sigma_j s)$ if $\beta_j \leqslant x < \beta_{j+1}$, $j \neq t$,
$T(x, s) = (x, \sigma_t s)$ if $\beta_t \leqslant x < 1 - \alpha$,
$T(x, s) = (x, \tau s)$ if $1 - \alpha \leqslant x < \beta_{t+1}$.

Non-triviality conditions : $\sigma_j \neq \sigma_{j+1}$, $0 \leqslant j \leqslant r - 1$, $j \neq t$ ; $\tau \neq \sigma_{t+1}$ ; $\sigma_t \sigma_r \neq \tau \sigma_0$.

# SYMBOLIC SYSTEMS

Symbolic system = the shift on infinite sequences on a finite alphabet.

Trajectories = $y_n = s_i$ if $T^n y$ falls into the $i$-th interval in the $s$-th copy of $[0, 1[$.

A trajectory of $T$ gives a trajectory of $R : u \to \phi(u)$ by $s_i \to i$, for all $i, s$.

Linear recurrence of the coding = in the language of trajectories of $R$, every word of length $n$ occurs in every word of length $Kn$.

## FIRST RESULT

**Theorem 1.** *Under the non-triviality conditions and minimality, $T$ is rigid (for any ergodic invariant measure) if $\alpha$ has unbounded partial quotients, $T$ is uniquely ergodic and non-rigid if the coding of $R$ by the partition determined by $\beta_1, ..., \beta_t, 1 - \alpha, \beta_{t+1}, ..., \beta_r$ is linearly recurrent.*

Proofs as for square-tiled iet :

— an iet with LR behaves as a rotation with BPQ,

— when the non-triviality conditions are satisfied on every letter (ex : Veech 1969), $T$ satisfies $\bar{d}$-separation,

— otherwise, average $\bar{d}$-separation.

# MINIMALITY

For Veech 1969, it was known that if $\beta$ is not in $\mathbb{Z}(\alpha)$, $T$ is minimal.

**Theorem 2.** *$T$ is minimal if and only if*

$$1 \pm \beta = 2m\alpha + 2n$$

*for some $m \in \mathbb{Z}$, $n \in \mathbb{Z}$.*

For the generalizations, we take all the $\beta_i$ and $\beta_i - \beta_j$ not in $\mathbb{Z}(\alpha)$.

**Proposition 1.** *An NCS for minimality is that no strict subset of $\{1 \dots d\}$ is invariant by all the $\sigma_i$ and $\tau$.*

# THE GREY ZONE AND OSTROWSKI

Case where $\alpha$ has bounded partial quotients but the coding of $R$ is NOT linearly recurrent.

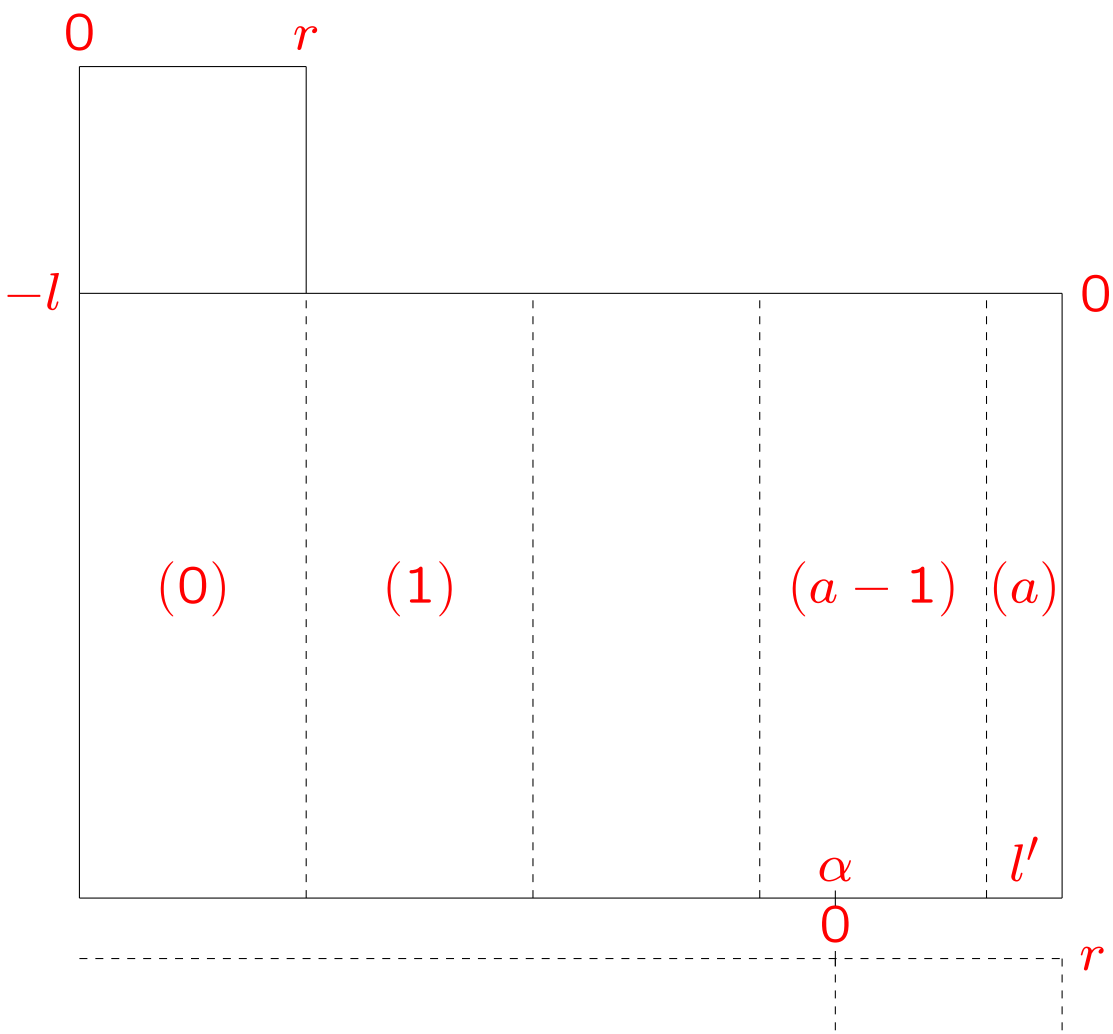
Let $\alpha$ be given, $a_n$ its partial quotients : we use a form of alternating Ostrowski expansion of each $\beta_i$ by $\alpha$, giving integers $0 \leqslant b_n(\beta_i) \leqslant a_n$. The Markov condition is $b_n = a_n$ implies $b_{n+1} = 0$.

The integer $b_{n+1}(\beta_i)$ tells us in which column is $\beta_i$ for the $n$-th Rokhlin tower of the rotation $R$.

# ROKHLIN TOWERS

$n$ is fixed $\quad a = a_{n+1} \quad r = |q_n\alpha - p_n| \quad l = |q_{n-1}\alpha - p_{n-1}|$ $l' = |q_{n+1}\alpha - p_{n+1}|$

Then, up to the symmetry $x \to -x$, we have two towers.



## BPQ BUT NOT LR

**Proposition 2.** *For $\alpha$ with BPQ, the coding of $R$ is NOT LR if and only if*

— *either there exists $i$ and pairs $M, N$ with $N - M$ arbitrarily large and $b_m(\beta_i) = a_m - 1$ for $M \leqslant m \leqslant N$,*

<u>*$\beta_i$ is close to $\alpha$*</u>

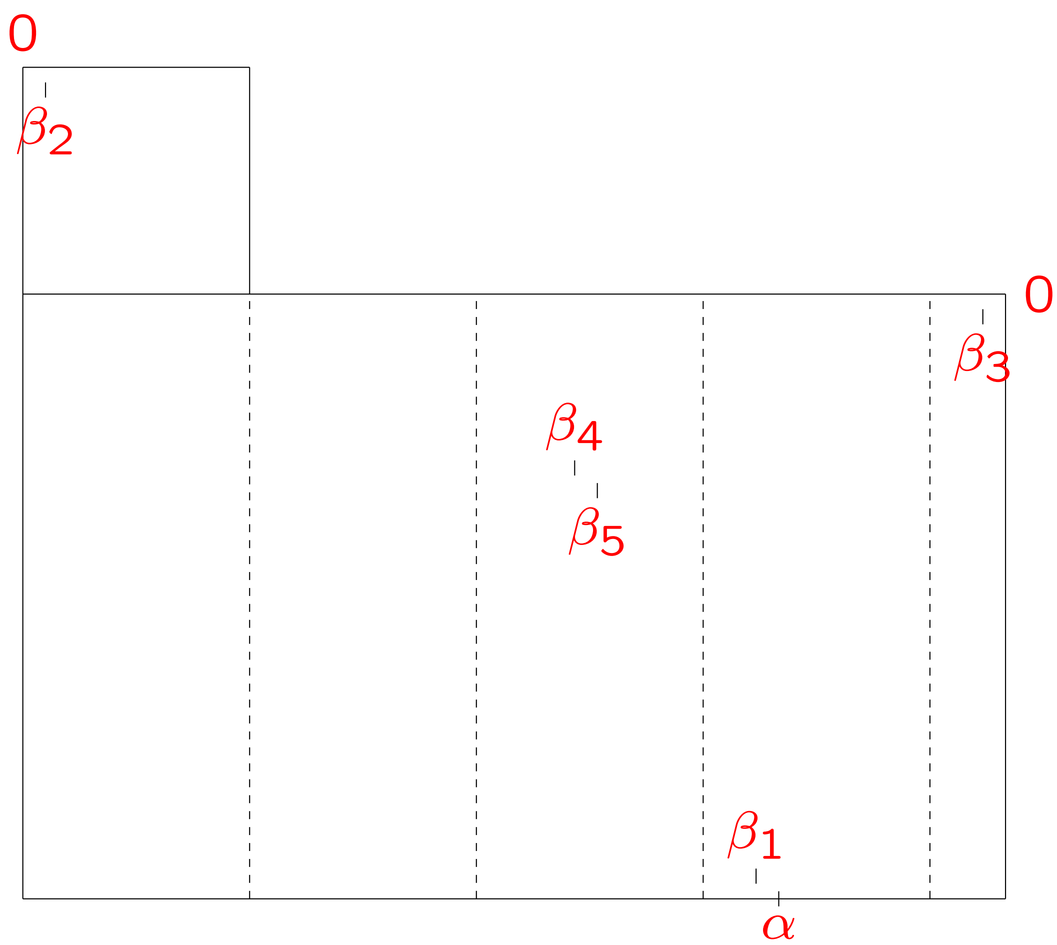
— *or there exists $i$ and pairs $M, N$ with $N - M$ arbitrarily large and $(b_m(\beta_i), b_{m+1}(\beta_i)) = (a_m, 0)$ for each $M \leqslant m = M + 2p \leqslant N$,*

<u>*$\beta_i$ is close to $\alpha$*</u> *(via $0$)*

— *or there exist $i \neq j$ and pairs $M, N$ with $N - M$ arbitrarily large and $b_m(\beta_j) = b_m(\beta_i)$ for $M \leqslant m \leqslant N$.*

<u>*$\beta_i$ is close to $\beta_j$*</u>

## *IN THE GREY ZONE*



## *NON-RIGID NON-LR*

**Theorem 3.** *If, whenever there is a run of $m$ as in Proposition 1, either no $\beta_i$ comes close to $\alpha$, or there exists $\beta_j$ such that no $\beta_k$, $k \neq j$, comes close to $\beta_j$, then $T$ is not rigid.*

*This gives the first examples of not rigid not LR iet.*

## *SO MANY SHADES OF GREY*

**Theorem 4.** *Suppose $\sigma_k\sigma_j = \sigma_j\sigma_k$ for all $j, k$, $\tau = \sigma_t$. If infinitely often all the $\beta_i$ come close to $\alpha$ at the same time, then $T$ is rigid.*

*This applies to Veech 1969, where $d = 2$ : in the grey zone $T$ is rigid.*

**Proposition 3.** *If there exist pairs $M, N$ with $N - M$ arbitrarily large, such that for all $i$ either $b_m(\beta_i) = a_m - 1$ for $M \leqslant m \leqslant N$, or $(b_m(\beta_i), b_{m+1}(\beta_i)) = (a_m, 0)$ for each even $M \leqslant m \leqslant N$, $a_{N-1} \neq 1$ and $a_N \neq 1$, then $T$ is rigid.*

*(Work in progress). When infinitely often all the $\beta_i$ come close to $\alpha$ at the same time, if Theorem 3 or 4 do not apply, and $\alpha$ is the golden ratio, we can build rigid and non-rigid examples.*

# *RIGID LR*

*An interval exchange with a circular permutation is a rotation, thus is rigid, even if it is LR.*

*The Arnoux-Yoccoz interval exchange is semi-conjugate to a rotation of the* $2$*- torus, which is rigid.*

*The measure-theoretic isomorphism between Arnoux-Yoccoz and the rotation, announced by Arnoux - Bernat - Bressaud (2011), was proved by Cassaigne (2018).*

*Thus Arnoux - Yoccoz is LR (self-induced, even) and rigid.*