Relations on word	s Synchronized relations	Class Containment Problem	The proof	Conclusions
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Resynchronizing Classes of Word Relations

María Emilia Descotte 1 LaBRI

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Languages

Relations

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Languages

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• Finite monoids







Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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LanguagesRelations• Finite monoids• REC• NFA's• REG

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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LanguagesRelations• Finite monoids \checkmark • REC \Diamond • NFA's \checkmark • Regular expressions \checkmark • Regular expressions \checkmark

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Relations on	words	Synchro	onized relation	s Class Contai	nment Problem	The proof	Conclusions	
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Synchronized pairs of words (over a fixed alphabet \mathbb{A})

Synchronizing pairs of words

A synchronization of (w_1, w_2) is a word over $\mathbf{2} \times \mathbb{A}$ so that the projection on \mathbb{A} of positions labeled *i* is exactly w_i for i = 1, 2.

	Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Example

(1, a)(1, b)(2, a) and (1, a)(2, a)(1, b)synchronize (ab, a).

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Every word $w \in (\mathbf{2} \times \mathbb{A})^*$ is a synchronization of a unique pair (w_1, w_2) that we denote $\llbracket w \rrbracket$.

 $[\![(1,a)(1,b)(2,a)]\!] = [\![(1,a)(2,a)(1,b)]\!] = (ab,a).$

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Synchronized	relations			

Synchronizing relations

We lift this notion to languages $L \subseteq (\mathbf{2} \times \mathbb{A})^*$

 $[\![L]\!] = \{[\![w]\!] \mid w \in L\}$

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Synchronized	relations			

Synchronizing relations

We lift this notion to languages $L \subseteq (\mathbf{2} \times \mathbb{A})^*$

 $[\![L]\!]=\{[\![w]\!]\mid w\in L\}$

Example

$$\mathbb{A} = \{a, b\}, \ L = ((1, a)(2, a) \cup (1, a)(2, b) \cup (1, b)(2, a) \cup (1, b)(2, b))^*,$$
$$\llbracket L \rrbracket = \{(w_1, w_2) \mid |w_1| = |w_2|\}.$$

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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C-controlled relations

Restrictions on the shape of the projection over $\mathbf{2}$

Infinitely many different classes of relations.

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions	
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C-controlled	relations				

Restrictions on the shape of the projection over 2

Infinitely many different classes of relations.

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C-controlled words and languages $C \subseteq \mathbf{2}^*$ regular $-w \in (\mathbf{2} \times \mathbb{A})^*$ is C-controlled if its projection over **2** belongs to C. $-L \subseteq (\mathbf{2} \times \mathbb{A})^*$ is C-controlled if all its words are.

	Relations on words O	Synchronized relations $\circ \circ \bullet$	Class Containment Problem 000	The proof 00000	Conclusions 00		
C	-controlled	relations					
	Restrictions on the shape of the projection over 2 $\sum_{i=1}^{i}$ Infinitely many different classes of relations.						
C	-controlled word	ls and languages	Examples				
-u pi -1	rojection over 2	C-controlled if its belongs to C . C-controlled if all	-Every $w \in (2 \times \mathbb{A})$ - $(1, a)(1, b)(2, a)$ is - $(1, a)(2, a)(1, b)$ is - L (previous slide)	1*2*-con n't 1*2*-c	trolled, controlled,		

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	Relations on words O	Synchronized relations $\circ \circ \bullet$	Class Containment Problem 000	The proof 00000	Conclusions 00			
(C-controlled relations							
	Restrictions on the shape of the projection over 2							
	$\sum_{i=1}^{i}$ Infinitely many different classes of relations.							
	C-controlled word	ls and languages	Examples					
	$C \subseteq 2^*$ regular $-w \in (2 \times \mathbb{A})^*$ is (projection over 2 $-L \subseteq (2 \times \mathbb{A})^*$ is (its words are.		-Every $w \in (2 \times \mathbb{A})$ - $(1, a)(1, b)(2, a)$ is - $(1, a)(2, a)(1, b)$ is - L (previous slide)	1*2*-con n't 1*2*-c	trolled, controlled,			

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C-controlled relations

Given a regular language $C \subseteq \mathbf{2}^*$

$$\operatorname{Rel}(C) = \left\{ \llbracket L \rrbracket \mid L \text{ is reg. and } C \text{-controlled} \right\}$$

Relations on words O	Synchronized relations $\circ \circ \bullet$	Class Containment Problem 000	The proof 00000	Conclusions 00		
C-controlled	relations					
Restrictions on the shape of the projection over 2						
Ι	nfinitely many diffe	$\stackrel{\Psi}{\operatorname{erent}}$ classes of relatio	ons.			
C-controlled word	ls and languages	Examples				
$C \subseteq 2^*$ regular $-w \in (2 \times \mathbb{A})^*$ is C projection over 2 $-L \subseteq (2 \times \mathbb{A})^*$ is C its words are.	belongs to C .	-Every $w \in (2 \times \mathbb{A})$ - $(1, a)(1, b)(2, a)$ is - $(1, a)(2, a)(1, b)$ is - L (previous slide)	1^{2} 1*2*-con n't 1*2*-c	trolled, controlled,		
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controlled relations

Given a regular language $C \subseteq \mathbf{2}^*$

$$\operatorname{Rel}(C) = \{ \llbracket L \rrbracket \mid L \text{ is reg. and } C \text{-controlled} \}$$

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-\text{Rel}(1^*2^*) = \text{REC},
-REL((12)^*(1^* \cup 2^*)) =REG,
-\operatorname{Rel}(2^*) = \operatorname{RAT}.
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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Class Containment Problem

CLASS CONTAINMENT PROBLEM Input: Two regular languages $C, D \subseteq \mathbf{2}^*$ Output: Is $\operatorname{Rel}(C) \subseteq \operatorname{Rel}(D)$?

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Class Containment Problem

CLASS CONTAINMENT PROBLEM Input: Two regular languages $C, D \subseteq \mathbf{2}^*$ Output: Is $\operatorname{Rel}(C) \subseteq \operatorname{Rel}(D)$?

Examples

-If
$$C \subseteq D$$
, then $\operatorname{REL}(C) \subseteq \operatorname{REL}(D)$,
- $\operatorname{REL}(1^*2^*) \subseteq \operatorname{REL}((12)^*(1^* \cup 2^*))$,
- $\operatorname{REL}((12)^*(1^* \cup 2^*)) \not\subseteq \operatorname{REL}(1^*2^*)$,
- $\operatorname{REL}(1^*2^*) = \operatorname{REL}(2^*1^*)$,
- $\operatorname{REL}((12)^*) = \operatorname{REL}((21)^*)$.

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions				
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Decidability and complexity

The problem is decidable for Rel(D) = REC, REG or Length-pres.

²D. Figueira and L. Libkin. Synchronizing relations on words. ACMTransactions on Computer Systems, 2015.

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Decidability and complexity

The problem is decidable for Rel(D) = REC, REG or Length-pres.

Resynchronization

The proof is constructive in terms of the automaton:

Given a C-controlled language L, one can effectively construct a D-controlled language L' such that [L] = [L'].

²D. Figueira and L. Libkin. Synchronizing relations on words. ACMTransactions on Computer Systems, 2015.

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Our contribution

We prove that the Class Containment Problem is decidable for arbitrary C and D and, in case of positive answer, we give an effective method for resynchronizing relations.

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Proof idea

Step 1: Rewrite C and D as finite unions of *simple languages*.

Step 2: Characterization for simple languages.

Step 3: Induction on the amount of disjuncts in the unions.

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Concat-star languages

$$C_1^* u_1 \cdots C_n^* u_n$$

with C_1, \ldots, C_n regular languages, u_1, \ldots, u_n words.



Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Concat-star languages

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with C_1, \ldots, C_n regular languages, u_1, \ldots, u_n words.

Simple languages

Concat-star languages of star-height 1 + extra restrictions.

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Simple languages

Concat-star languages of star-height 1 + extra restrictions.

Examples • $1^*(12)^*2^*12 \checkmark$ • $(1^*2)^*2^*11 \nearrow$ • $1^*(12 \cup 1)^*(112)^*1 \checkmark$ • $(12)^*1^* \cup (12)^*2^* \bigstar$

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Every regular language is a finite union of concat-star languages.

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Every regular language is a finite union of concat-star languages.

Every concat-star language is *Rel-equivalent* to a finite union of concat-star languages of star-height 1.

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Every regular language is a finite union of concat-star languages.

Every concat-star language is *Rel-equivalent* to a finite union of concat-star languages of star-height 1.

Every concat-star language of star-height 1 is *Rel-equivalent* to a finite union of simple languages.

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Step 2: Chara	acterization f	or simple langua	ages	

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Parikh ratio

$$\begin{aligned} -w \in \mathbf{2}^* \setminus \{\varepsilon\}, \, \rho(w) &= \frac{|w|_1}{|w|}.\\ -C \subseteq \mathbf{2}^*, \, \rho(C) &= \{\rho(w) \mid w \in C \setminus \{\varepsilon\}\} \subseteq [0,1]_{\mathbb{Q}}. \end{aligned}$$

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Step 2: Characterization for simple languages

Parikh ratio

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Synchronizing morphisms

$$C = C_1^* u_1 \cdots C_n^* u_n, D = D_1^* v_1 \cdots D_m^* v_m. C \xrightarrow{s.m.} D$$
 is

$$f:[1,\ldots,n] \to [1,\ldots,m]$$
 s.t.

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i) f is monotonic and ii) $\rho(C_i^*) \subseteq \rho(D_{f(i)}^*)$ for all $i = 1, \dots, n$.

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Step 2: Characterization for simple languages

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 s.t.

i)
$$f$$
 is monotonic and
ii) $\rho(C_i^*) \subseteq \rho(D_{f(i)}^*)$ for all $i = 1, \dots, n$.

$$\begin{array}{c} 2^{*} 1^{*} (122 \cup 12)^{*} (122)^{*} (112)^{*} 1^{*} 2^{*} (22)^{*} \\ f \\ (22)^{*} 1^{*} (122 \cup 112)^{*} (11 \cup 111)^{*} (12)^{*} 2^{*} \end{array}$$

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Step 2: Chara	acterization for	or simple langua	ages	

Proposition

For all simple languages $C, D \subseteq \mathbf{2}^*$,

 $\operatorname{Rel}(C) \subseteq \operatorname{Rel}(D) \text{ iff } \pi(C) \subseteq \pi(D) \text{ and } C \xrightarrow{s.m.} D.$



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Step 2: Ch	aracterization	for simple langu	ages	

Proposition

For all simple languages $C, D \subseteq \mathbf{2}^*$,

 $\operatorname{Rel}(C) \subseteq \operatorname{Rel}(D)$ iff $\pi(C) \subseteq \pi(D)$ and $C \xrightarrow{s.m.} D$.

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Examples

 $-\operatorname{ReL}((12)^*(112)^*) \subseteq \operatorname{ReL}((12 \cup 11122)^*(121)^*1^*2^*), \\ -\operatorname{ReL}((112)^*(12)^*) \not\subseteq \operatorname{ReL}((12 \cup 11122)^*(121)^*1^*2^*).$

Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Step 3: Dealing with unions

Unions on the left

$\operatorname{Rel}(C_1 \cup C_2) \subseteq \operatorname{Rel}(D) \text{ iff}$ $\operatorname{Rel}(C_1) \subseteq \operatorname{Rel}(D) \text{ and } \operatorname{Rel}(C_2) \subseteq \operatorname{Rel}(D).$

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Step 3: Dealing with unions

Unions on the left

 $\operatorname{Rel}(C_1 \cup C_2) \subseteq \operatorname{Rel}(D) \text{ iff}$ $\operatorname{Rel}(C_1) \subseteq \operatorname{Rel}(D) \text{ and } \operatorname{Rel}(C_2) \subseteq \operatorname{Rel}(D).$

Unions on the right

For C simple and $D = \bigcup_j D_j$ a finite union of simple languages, the following are equivalent:

- i) $\operatorname{Rel}(C) \subseteq \operatorname{Rel}(D)$,
- **ii)** $\pi(C) \subseteq \pi(D), \exists j \text{ with } C \xrightarrow{s.m.} D_j \text{ and in addition, if } C \text{ is heterogeneous, then } \operatorname{REL}(C \setminus [D_j]_{\pi}) \subseteq \operatorname{REL}(\bigcup_{j' \neq j} D_{j'}).$

$$[D_j]_{\pi} = \pi^{-1}(\pi(D_j)) = \{ w \mid \pi(w) \in \pi(D_j) \}.$$

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Future work

• Our proof gives an effective algorithm to resynchronize relations. We would like to determine the exact complexity.

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Future work

- Our proof gives an effective algorithm to resynchronize relations. We would like to determine the exact complexity.
- What about k-ary relations? Step 1 relies on geometric arguments that only hold in dimension 2.

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Relations on words	Synchronized relations	Class Containment Problem	The proof	Conclusions
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Thanks for your attention!

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