A definition and counting of biperiodic recurrent configurations in the sandpile model on $\mathbb{Z}^2$

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LaBRI

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Definition of the model [Bak, Tang and Wiesenfeld, 1987].

We study the diffusion of the red quantity in the blue rectangle.

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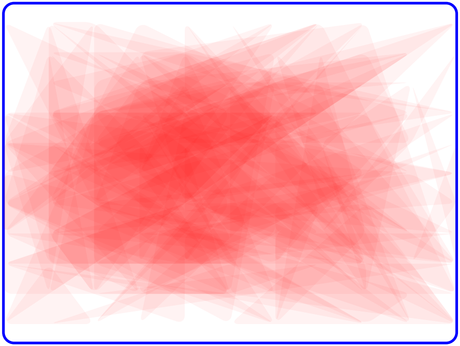

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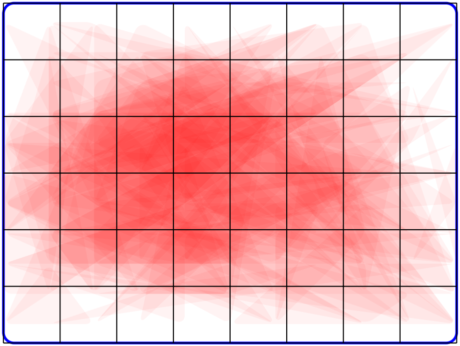

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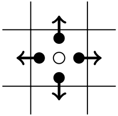

$$u \to u + \Delta_{v_e}$$

Vector coding the grains transfer:

$\Delta_{v_e} = \ldots 1, \ldots 1, -4, 1, \ldots, 1, \ldots$

Laplacian matrix: $\Delta = (\Delta_v)_{v\in\mathbb{Z}^2}$

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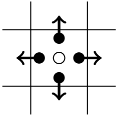

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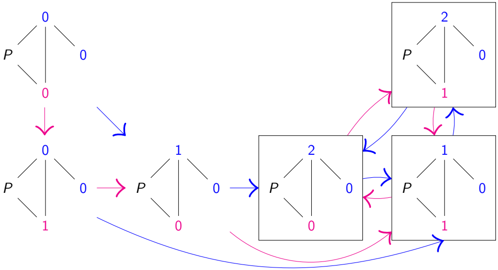
Stabilisation: While there is a unstable vertex, we topple it.

We note stab(u) the potential stable configuration resulting from the stabilisation.

Markov Chain for $G = (V \cup \{S\}, E)$

- States: stable configurations on G
- Transition: Add a particle to a vertex chosen uniformly and stabilize

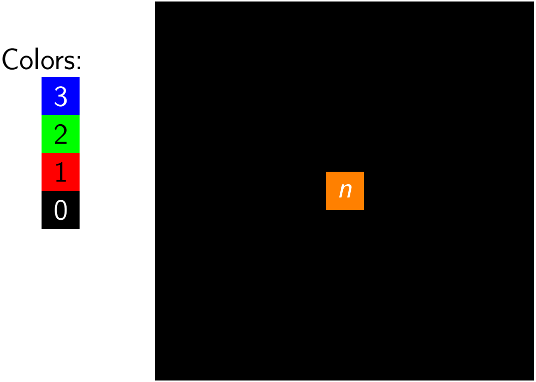


- The recurrent states are called recurrent configurations.
- The stationary distribution is uniform on the recurrent configurations.

Dhar Criterion A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)

Self-organized criticality in the sand pile on $\mathbb{Z}^2$

Let $\delta^{(n)}$ be the configuration such that $\delta^{(n)}_{(0,0)} = n$ and $\delta^{(n)}_{(i,j)} = 0$ otherwise.



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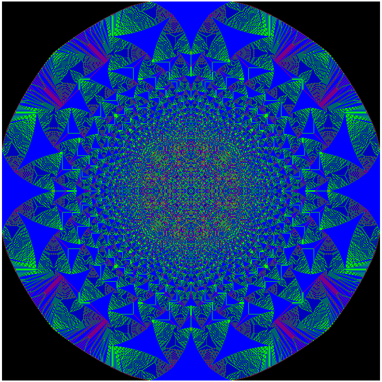
Colors:
3
2
1
0



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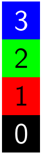

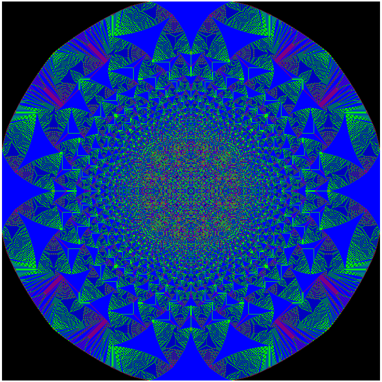

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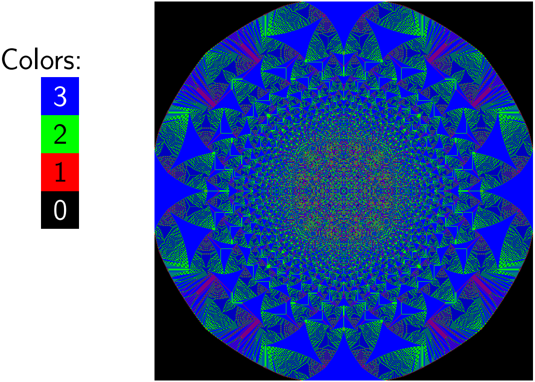

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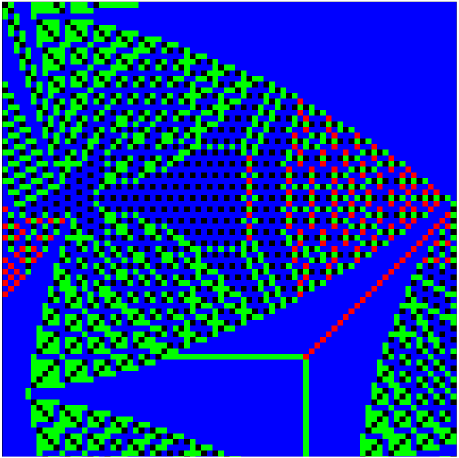
- Size $O(\sqrt{n})$
- Fractal structure

Periodic pattern in zones



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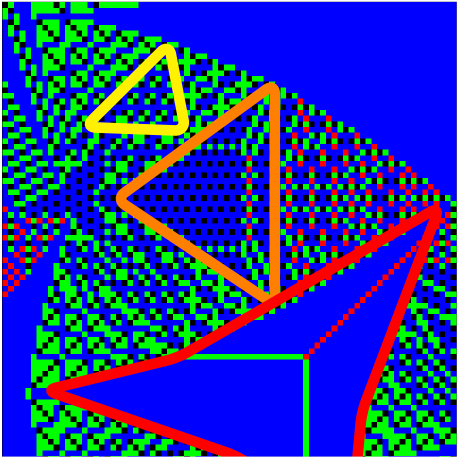
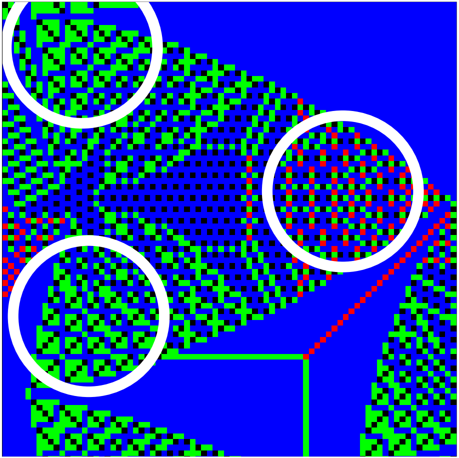


Periodic pattern in zones

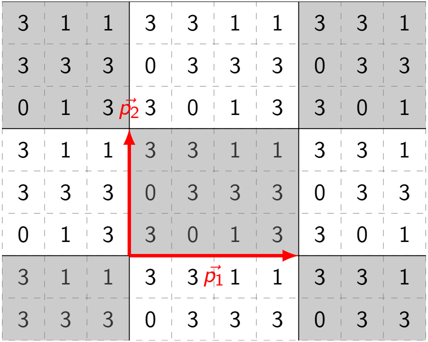
- Triangular pieces periodic with 1D imperfections
- Periodic pieces behave in some way like recurrent configuration



A definition of recurrence for periodic stable configurations

Pattern + two dimensional period $(\vec{p_1}, \vec{p_2})$.

$$\forall \mathbf{x} \in \mathbb{Z}^2 u(\mathbf{x}) = u(\mathbf{x} + \vec{p_1}) = u(\mathbf{x} + \vec{p_2})$$



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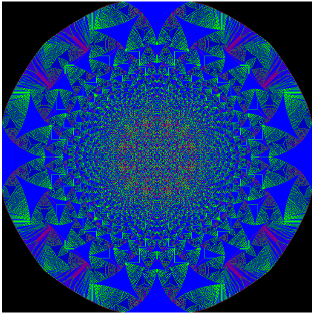

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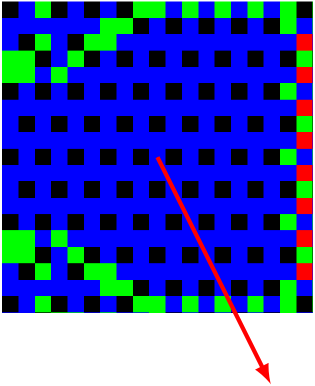

- On an arbitrary vertex $\Rightarrow$ Singularity.
- Far from every vertex $\Rightarrow$ To the infinity in some direction.

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Definition

A periodic configuration is recurrent in the direction $\mathbf{s} \in \mathbb{Z}^2$ if for any $c \in \mathbf{R}$, forcing the toppling of the half plan $\{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{s} > c\}$ leads by stabilisation to the toppling of each vertex exactly once.

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0	1	3	3	0	1	3	3	0	1
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This criterion can be computed in finite time.

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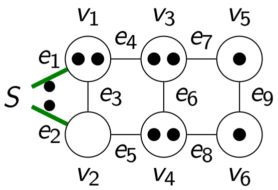
- Ordering the toppling roughly by distance to the half-plan
- Remark periodicity along $\mathbf{s}^{\perp} \Rightarrow$ Simplify the graph to a cylinder
- Criterion is ultimately periodic along $-\mathbf{s}$

Bijections between recurrent configurations and spanning trees

Some bijections : Dhar/Majumdar 92, Cori/Le Borgne 03, Bernardi 06...
We choose an order on the edges: from left to right and from top to bottom. $e_1 \succ e_2 \succ e_3 \ldots$.

Bijection CLB: (from a recurrent configuration)
Decreasing edge-vertex traversals:

$e_1, v_1, e_2, e_3, e_4, v_3, e_6, v_4, e_5, v_2, e_7, v_5, e_8, v_6, e_9$

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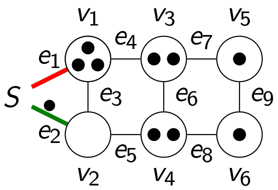


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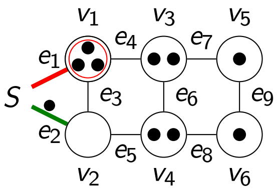


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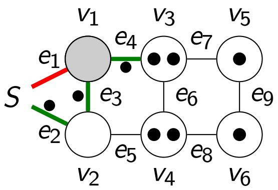


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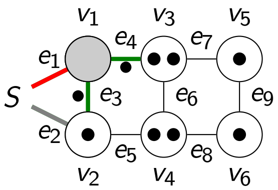


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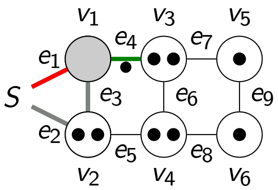


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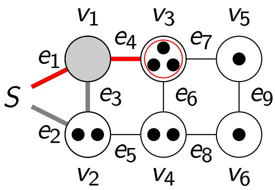


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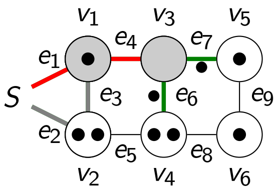


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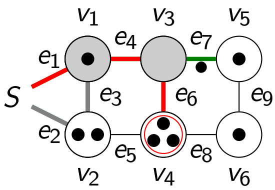


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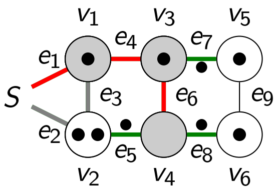


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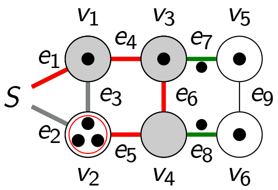


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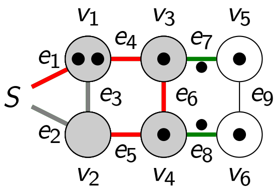


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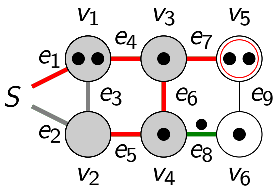


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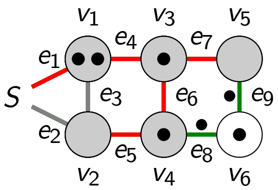


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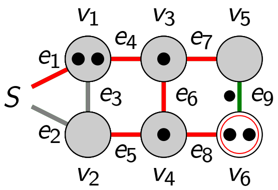


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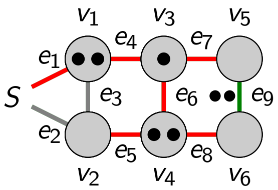


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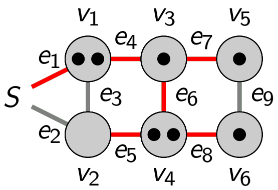


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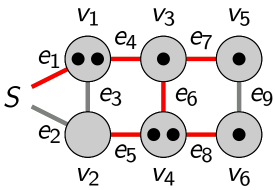


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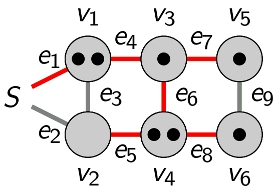


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Periodicity on $-\mathbf{s}^\perp$

Let e_1 and e_2 two edges, m_1 and m_2 their middle.

$$e_1 \prec_{\mathbf{s}} e_2 \Leftrightarrow \mathbf{s} \cdot \mathbf{m_1} < \mathbf{s} \cdot \mathbf{m_2} \text{ or } (\mathbf{s} \cdot \mathbf{m_1} = \mathbf{s} \cdot \mathbf{m_2} \text{ et } \mathbf{s}^\perp \cdot \mathbf{m_1} < \mathbf{s}^\perp \cdot \mathbf{m_2}).$$

For $\mathbf{s} = (0, -1)$, the largest edge is the one at the bottom left.
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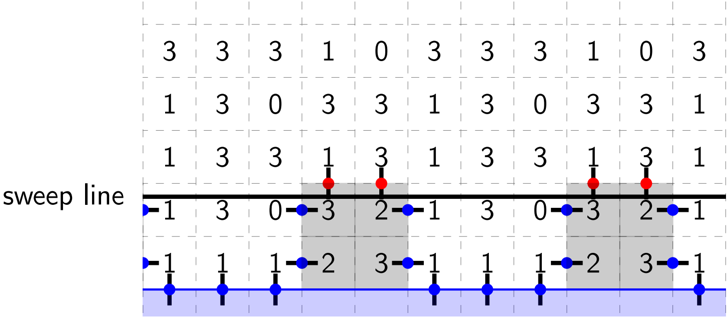

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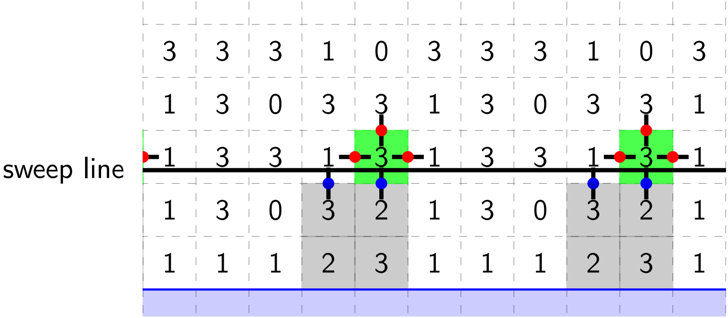

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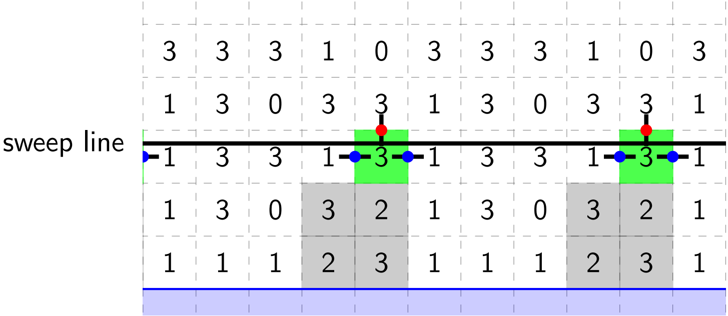

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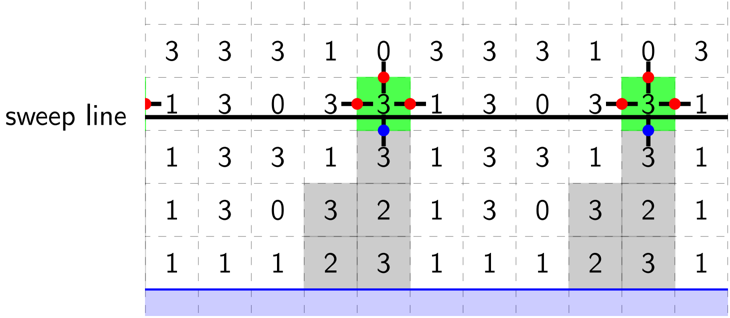

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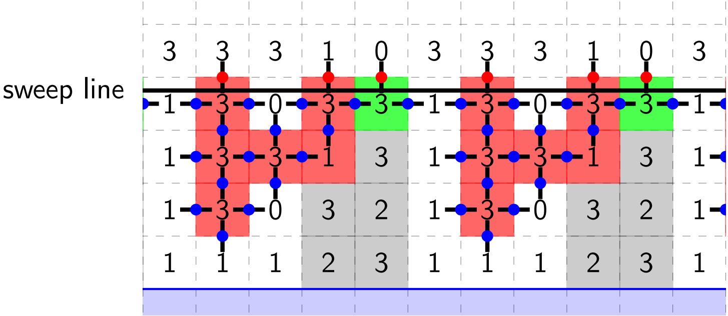
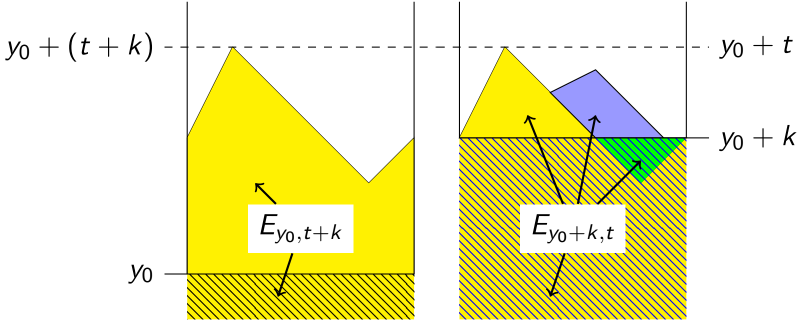


Periodicity on $-\mathbf{s}$

We force the toppling of the half-plan $y < y_0$. $E_{y_0,t}$ is the set of toppled vertices when the sweep line is at position $y = t$. (Note: $\{(x, y) \mid y < y_0\} \subset E_{y_0,t}$)



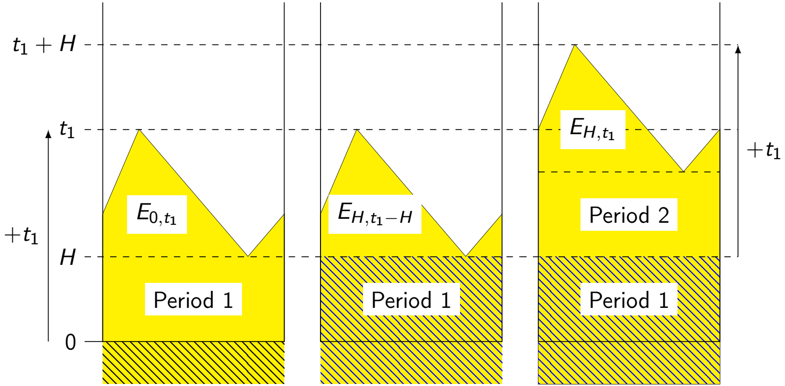
Then

$$E_{y_0,t+k} \subset E_{y_0+k,t}$$

Periodicity on $-\mathbf{s}$

Lemme

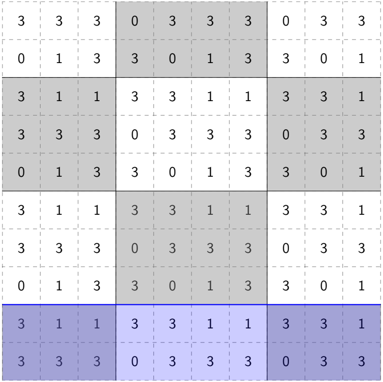
If a periodic configuration is recurrent, then there exists a position $y = t_1$ for which all vertices of the first period are toppled.



We have $Period1 \subset E_{0,t_1} \Rightarrow E_{0,t_1} = E_{H,t_1-H}$ and $v \in E_{0,t_1} \Rightarrow v + H\vec{y} \in E_{H,t_1}$. Then $E_{H,t_1} \supset Period2$.

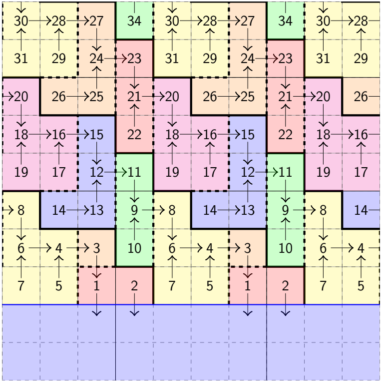
Bijection: periodic covering and recurrent

Using the trace of the criterion, we extract a periodic spanning forest.



$x = 0$

$y = 0$



Tree from the criterion

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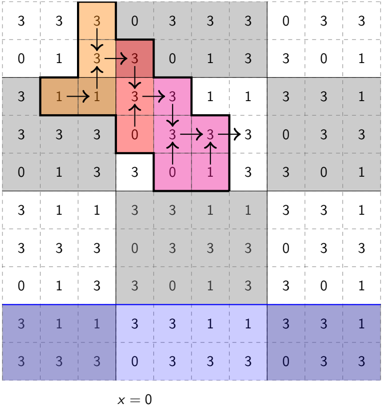

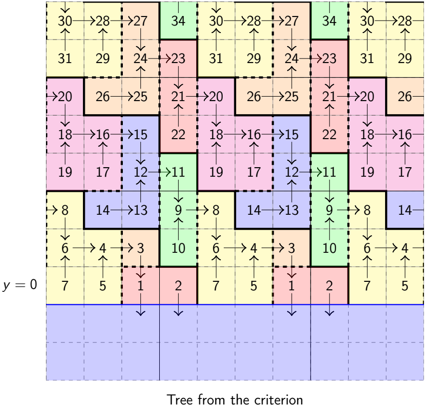

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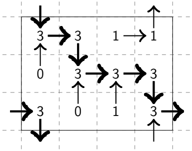

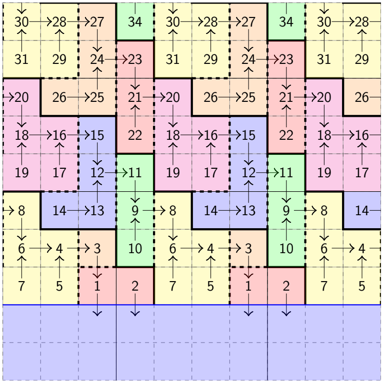

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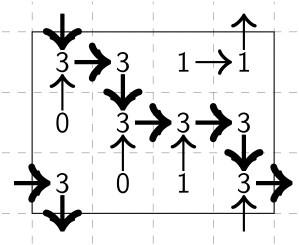

On the torus,

- Each vertex has a father

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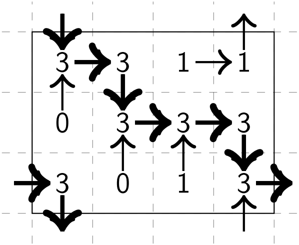

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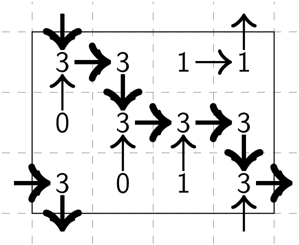

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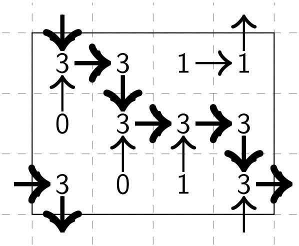

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Theorem

Given a direction $\mathbf{s}$*, the periodic configuration recurrent in the direction* $\mathbf{s}$ *and the (undirected) periodic spanning forests of infinite branches with slopes non collinear to* $\mathbf{s}^{\perp}$ *are in bijection.*

Some results about periodic configurations

Counting

Counting of periodic spanning forest without contractible cycles by a determinantal formula [Kenyon].
Filtering by slope $\Rightarrow$ Counting of periodic recurrent configuration in a direction.

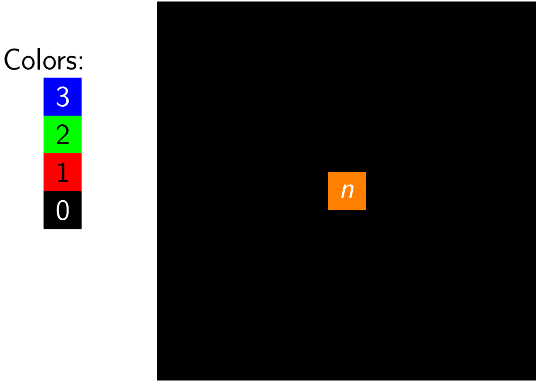
A group on the periodic recurrent configurations ?

On finite graph, recurrent configurations equipped with $u, v \mapsto stab(u+v)$ is an abelian group.
Is it still possible with infinite configurations? Periodic configurations? How to stabilize? (WIP) Related to quadratic forms.

The bijection enhances the analysts approximation by quadratic forms for the periodic pieces one the sand pile with n grains.

Self-organized criticality in the sand pile on $\mathbb{Z}^2$

Let $\delta^{(n)}$ be the configuration such that $\delta^{(n)}_{(0,0)} = n$ and $\delta^{(n)}_{(i,j)} = 0$ otherwise.



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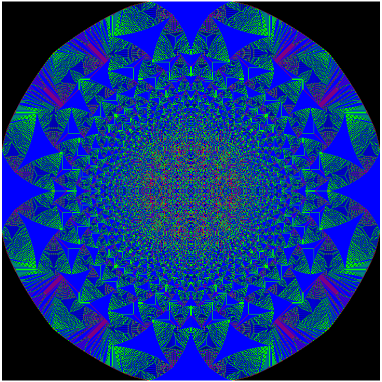
0



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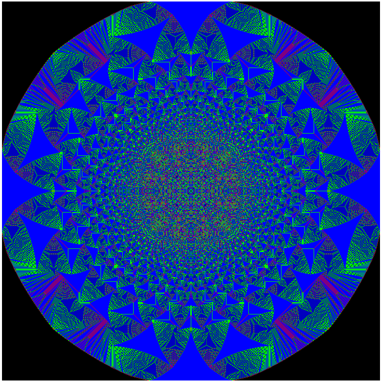

The physicists Creutz and Bak [circa 90] relying on experimental results:
A fractal structure.

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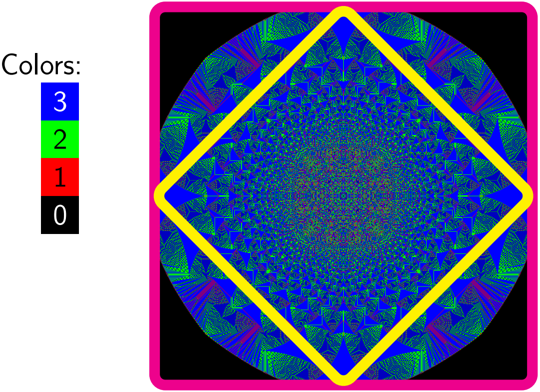

The computer scientist Rossin and Le Borgne [circa 00] show bounds on the domain by two squares, then the domain's radius has order of $\sqrt{n}$.

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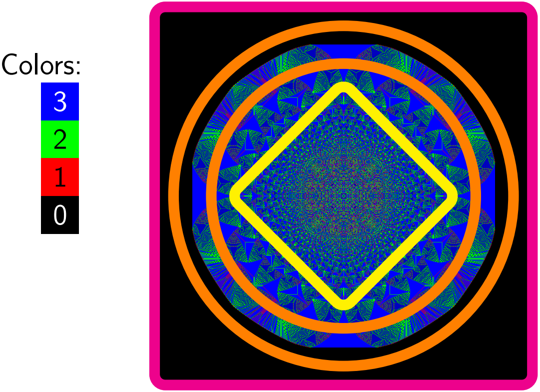

The probabilists Levine and Peres [circa 2007] show that the the frontier of the domain of the $(\text{stab}(\delta^{(n)}))_{n\in\mathbb{N}}$, rescaled by $1/\sqrt{n}$, stays between to two concentric circle of radius $1/\sqrt{k\pi} + o(n)$ for $k \in \{2, 3\}$.

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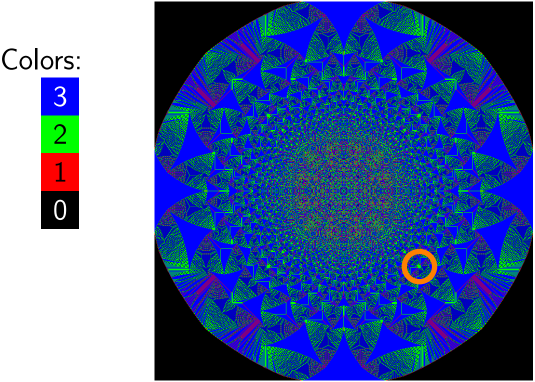


The analysts Pegden and Smart [circa 2012] show that the set of configurations converges “in term of viscosity”. (The density of grains in a zone of the ”limit object” is well defined)

Close to [Pegden and Smart, 2017]

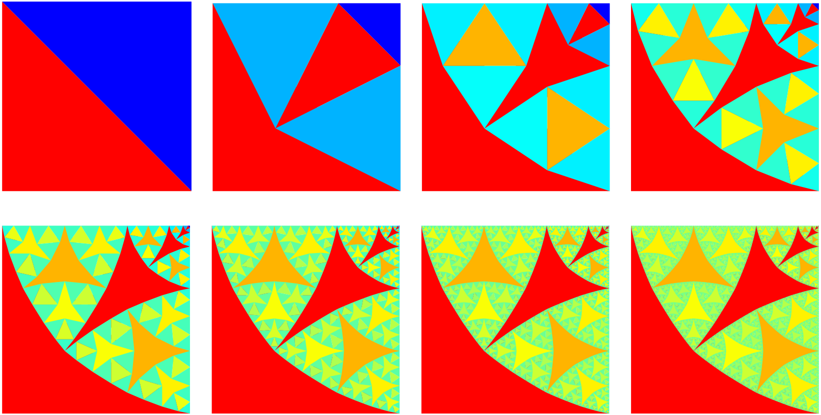


Figure: Each non blue zone is described by a quadratic form. [arxiv:1708.09432]

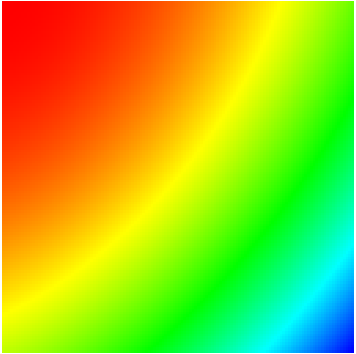
Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

$$M(a,b,c) = \begin{pmatrix} c+a & b \\ b & c-a \end{pmatrix}$$

The number of topples is:

$$h(\mathbf{x}) = \left\lceil \frac{1}{2}\mathbf{x}^t M(a,b,c)\mathbf{x} \right\rceil$$
$$= (c+a)x^2 + 2bxy + (c-a)y^2$$

Sample for $M(0.25, 0.875, 2.125)$



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Then number of grains is

$$\Delta h(\mathbf{u}) = \sum_{\mathbf{v}\sim\mathbf{u}} h(\mathbf{v}) - h(\mathbf{u}).$$

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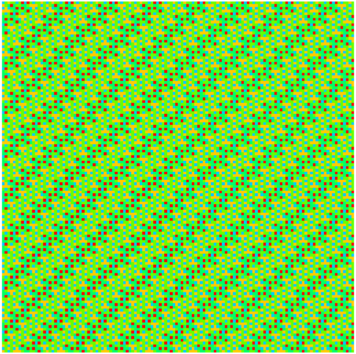

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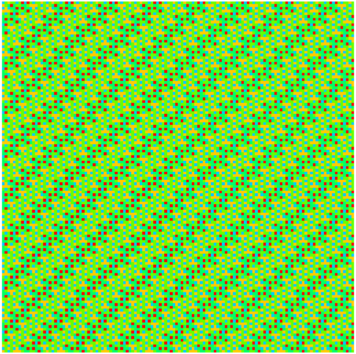
$$h(\mathbf{x}) = \left\lceil \frac{1}{2}\mathbf{x}^t M(a,b,c)\mathbf{x} \right\rceil$$
$$= (c+a)x^2 + 2bxy + (c-a)y^2$$

Then number of grains is

$$\Delta h(\mathbf{u}) = \sum_{\mathbf{v}\sim\mathbf{u}} h(\mathbf{v}) - h(\mathbf{u}).$$

- It's *periodic* for $a, b, c \in \mathbb{Q}$

Sample for $M(0.25, 0.875, 2.125)$



Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

$$M(a,b,c) = \begin{pmatrix} c+a & b \\ b & c-a \end{pmatrix}$$

The number of topples is:

$$h(\mathbf{x}) = \left\lceil \frac{1}{2}\mathbf{x}^t M(a,b,c)\mathbf{x} \right\rceil$$
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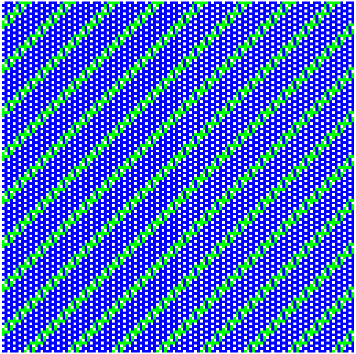
Sample for $M(0.25, 0.875, 2.125)$



- It's *periodic* for $a, b, c \in \mathbb{Q}$
- But it may be negative and/or unstable!

It may be stabilized without changing density of grains.

THANK YOU