Generalized Beatty sequences and complementary triples

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Joint work with Jean-Paul Allouche

17$^{\mathrm{e}}$ Journeés Montoises 25 minutes lecture

September 11, 2018—just before the wine tasting

Beatty sequences

Beatty sequence: $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$, where α is a positive real number.

Beatty observed: if $B(n) := \lfloor n\beta \rfloor$, with

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{1}$$

then $(A(n))$ and $(B(n))$ are *complementary* sequences.

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Example $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio.
$(\lfloor n\varphi \rfloor)_{n\geq 1}$ and $(\lfloor n\varphi^2 \rfloor)_{n\geq 1}$ are complementary.
$A = (1, 3, 4, 6, 8, \dots,), \quad B = (2, 5, 7, 10, 13, \dots).$

Carlitz-Scoville-Hoggatt

Consider the monoid generated by $(A(n))_{n\geq 1}$ and $(B(n))_{n\geq 1}$ for the composition of sequences.

Choose $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$.

Theorem (Carlitz-Scoville-Hoggatt)

Let $U = (U(n))_{n\geq 1}$ be a composition of the sequences $A = (\lfloor n\varphi \rfloor)_{n\geq 1}$ and $B = (\lfloor n\varphi^2 \rfloor)_{n\geq 1}$, containing i occurrences of A and j occurrences of B, then for all $n \geq 1$

$$U(n) = F_{i+2j}A(n) + F_{i+2j-1}n - \lambda_U,$$

where F_k are the Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$) and λ_U a constant.

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Example $B(B(A(n))) = 5A(n) + 3n - 3$ for all $n \geq 1$.

Generalized Beatty sequences

$(U(n)) = (F_{i+2j}A(n) + F_{i+2j-1}n - \lambda_U)$ is an example of a GBS.

Definition of generalized Beatty sequence V:

$V(n) = p\lfloor n\alpha \rfloor + qn + r,\ n = 1, 2, \ldots$ where p, q, r are integers.

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We also admit:

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Questions

Question 1 Let α be an irrational number, and let A defined by $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$ be the Beatty sequence of α.

Let Id defined by $\mathrm{Id}(n) = n$.

For which sixtuples of integers p, q, r, s, t, u are the two sequences

$$V = pA + q\,\mathrm{Id} + r \;\text{ and }\; W = sA + t\,\mathrm{Id} + u$$

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Question 2 For which 9-tuples of integers $(p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3)$ the three sequences

$$V_i = p_i A + q_i\,\mathrm{Id} + r_i, \; i = 1, 2, 3$$

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are a complementary triple?

Complementary triple: three sequences so that the sets they determine are disjoint with union the positive integers.

Intermezzo: Sturmian words

A Sturmian word w is an infinite word $w = w_0 w_1 w_2 \ldots$, in which occur only $n + 1$ subwords of length n for $n = 1, 2 \ldots$.

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$$w_n = s_{\alpha,\rho}(n) = [(n+1)\alpha + \rho] - [n\alpha + \rho], \quad n = 0, 1, 2, \dots.$$

or as

$$w_n = s'_{\alpha,\rho}(n) = \lceil (n+1)\alpha + \rho \rceil - \lceil n\alpha + \rho \rceil, \quad n = 0, 1, 2, \dots.$$

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Example $\alpha = \frac{1+\sqrt{5}}{2}, \rho = 0$. Here $w = 2122121221221212\ldots$, obtained by replacing 0 by 2 in the unique fixed point x_{F} of the Fibonacci morphism $0 \to 01$, $1 \to 0$.

How to recognize a golden mean GBS

$\alpha = \varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

Lemma

Let $V = (V(n))_{n \geq 1}$ *be the generalized Beatty sequence defined by* $V(n) = p(\lfloor n\varphi \rfloor) + qn + r$, *and let* ΔV *be the sequence of its first differences. Then* ΔV *is the Fibonacci sequence on the alphabet* $\{2p + q, p + q\}$.

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Proof:

$V(n+1)-V(n) = p\big[A(n+1)-A(n)\big]+q = p\big[\lfloor (n+1)\varphi\rfloor - \lfloor n\varphi \rfloor\big]+q.$

Partial answer to Question 1, part I

Theorem

Let $\alpha = \varphi$. Then there are no more than two increasing solutions with $V(1) = 1$ to the complementary pair problem:
$(p, q, r, s, t, u) = (1, 0, 0, 1, 1, 0)$ *and*
$(p, q, r, s, t, u) = (-1, 3, -1, 1, 2, 0)$.

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$(p, q, r, s, t, u) = (-1, 3, -1, 1, 2, 0)$.

The solutions are the two Beatty pairs

$([n\varphi]), ([n\varphi^2])$ and

$([n(3-\varphi)]), ([n(\varphi+2)])$.

Partial answer to Question 1, part II

Theorem

[Odd Fibonacci] *Let $\alpha = \varphi$. Any solution (p, q, r, s, t, u) to the complementary pair problem with $p > 0$ has to satisfy: p divides some Fibonacci number of odd index, i.e., p divides some number in the set $\{1, 2, 5, 13, 34, \ldots\}$.*

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Corollary

There are no solutions to the golden mean complementary pair problem if -1 is not a square modulo p, i.e., if p does not belong to the sequence $1, 2, 5, 10, 13, 17, 25, 26, 29, 34, 37, 41, \ldots$

Proof sketch of the "odd Fibonacci" Theorem

Consider the densities of V and W in $\mathbb{N}$ $\Rightarrow$ *necessary* condition for $(pA + q\,\mathrm{Id} + r, sA + t\,\mathrm{Id} + u)$ to be a complementary pair is that

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Lemma

Let $\alpha = \varphi$. A necessary condition for the pair $V = pA + q\,\mathrm{Id} + r$ and $W = sA + t\,\mathrm{Id} + u$ to be a complementary pair is that $p \neq 0$ is a solution to the generalized Pell equation

$$5p^2x^2 - 4x = y^2, \quad x, y \in \mathbb{Z}$$

.

Another example of GBS's

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A homomorphism of $\mathcal{L}$ into the natural numbers is a map $\mathrm{S} : \mathcal{L} \to \mathbb{N}$ satisfying $\mathrm{S}(vw) = \mathrm{S}(v) + \mathrm{S}(w)$, for all $v, w \in \mathcal{L}$.

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Let $\mathcal{L}_{\mathrm{F}}$ be the Fibonacci language, i.e., the set of all words occurring in $x_{\mathrm{F}} = 010010100100\dots$.

Theorem

[D.,TCS,2018]

Let $\mathrm{S} : \mathcal{L}_{\mathrm{F}} \to \mathbb{N}$ *be a homomorphism. Define* $a = \mathrm{S}(0), b = \mathrm{S}(1)$. *Then* $\mathrm{S}(\mathcal{L}_{\mathrm{F}})$ *is the union of the two generalized Beatty sequences* $\big((a-b)\lfloor n\varphi\rfloor + (2b-a)n\big)$ *and* $\big((a-b)\lfloor n\varphi\rfloor + (2b-a)n + a - b\big)$.

Homomorphisms and complementary triples

When is $\mathbb{N} \setminus \mathrm{S}(\mathcal{L}_\mathrm{F})$ also a GBS?

Actually this happens for only a few homomorphisms S!

Main example Let S be given by $\mathrm{S}(0) = 3$ and $\mathrm{S}(1) = 1$.
Then $\mathrm{S}(\mathcal{L}_\mathrm{F})$ is $(2\lfloor n\varphi \rfloor - n)_{n\geq 1} = 1, 4, 5, 8, 11, 12, 15, 16, \ldots.$
together with $(2\lfloor n\varphi \rfloor - n + 2)_{n\geq 1} = 3, 6, 7, 10, 13, 14, 17, 18, \ldots,$
and $\mathbb{N} \setminus \mathrm{S}(\mathcal{L}_\mathrm{F}) = 2, 9, 20, 27, 38, 49, \ldots.$

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$\mathrm{S}(11) = 2$, but $11 \not\prec \mathcal{L}_\mathrm{F}$.

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$\mathrm{S}(11) = 2$, but $11 \not< \mathcal{L}_\mathrm{F}$.

$\mathrm{S}(10101) = 9$, but $10101 \not< \mathcal{L}_\mathrm{F}$.

Theorem

$\mathbb{N} \setminus \mathrm{S}(\mathcal{L}_\mathrm{F}) = (4\lfloor n\varphi \rfloor + 3n + 2)_{n\geq 0}.$

Final example of GBS's

Theorem

Let x_{F} be the Fibonacci word, and let w be any word in the Fibonacci language $\mathcal{L}_{\mathrm{F}}$. Let Y be the sequence of positions of the occurrences of w in x_{F}. Then Y is a generalized Beatty sequence, i.e., for all $n \geq 0$ $Y(n+1) = p\lfloor n\varphi \rfloor + qn + r$ with parameters p, q, r, which can be explicitly computed.

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Proof idea: Let $x_{\mathrm{F}} = r_0(w)r_1(w)r_2(w)r_3(w)\ldots$, written as a concatenation of return words of the word w.

Main theorem in **[Huang & Wen, TCS, 2015]**:

if we skip $r_0(w)$, then the return words occur as the Fibonacci word on the alphabet $\{r_1(w), r_2(w)\}$.

THE END

Final treat: Creating triples from pairs

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Theorem

Let (V, W) *be a complementary pair* $V = pA + q\,\mathrm{Id} + r$ *and* $W = sA + t\,\mathrm{Id} + u$*. Then* (V_1, V_2, V_3) *is a complementary triple, where the three parameters of* V_1 *are* $(p+q, p, r-p)$*, those of* V_2 *are* $(2p+q, p+q, r)$*, and* $V_3 = W$*.*

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Proof idea:

$$A(\mathbb{N}) \cup B(\mathbb{N}) = \mathbb{N}, \quad V(\mathbb{N}) \cup W(\mathbb{N}) = \mathbb{N}.$$

Put one into the other $\Rightarrow$ you obtain the disjoint union

$$V(A(\mathbb{N})) \cup V(B(\mathbb{N})) \cup W(\mathbb{N}) = \mathbb{N}.$$

Carlitz-Scoville-Hoggatt Theorem $\Rightarrow$

$$AA(n) = A(n) + n - 1,\; AB(n) = 2A(n) + n.$$

Michel Dekking, Substitution invariant Sturmian words and binary trees, *Integers* **18A** (2018), #A7, 1-14.

J.-P. Allouche, B. Cloitre, V. Shevelev, Beyond odious and evil, *Aequationes Math.* **90** (2016), 341–353.

C. Ballot, On functions expressible as words on a pair of Beatty sequences, *J. Integer Seq.* **20** (2017), Art. 17.4.2.

L. Carlitz, R. Scoville, V. E. Hoggatt, Jr., Fibonacci representations, *Fibonacci Quart.* **10** (1972), 1–28. [Also see by the same authors: Addendum to the paper: "Fibonacci representations", *Fibonacci Quart.* **10** (1972), 527–530.

F. Michel Dekking, Morphisms, Symbolic Sequences, and Their Standard Forms, Journal of Integer Sequences, Vol. 19 (2016), Article 16.1.1.

F. M. Dekking, The Frobenius problem for homomorphic embeddings of languages into the integers, *Theoret. Comput. Sci.* **732** (2018), 73–79.

F. M. Dekking, Substitution invariant Sturmian words and binary trees, *Integers* **18A** (2018), #A7, 1-14.

A. S. Fraenkel, Complementary systems of integers, *Amer. Math. Monthly* **84** (1977), 114–115.

A. S. Fraenkel, Iterated floor function, algebraic numbers, discrete chaos, Beatty subsequences, semigroups, *Trans. Amer. Math. Soc.* **341** (1994), 639–664.

A. S. Fraenkel, Complementary iterated floor words and the Flora game, *SIAM J. Discrete Math.* **24** (2010), 570–588.

A. S. Fraenkel, From enmity to amity, *Amer. Math. Monthly* **117** (2010), 646–648.

Y. Huang, Z.-Y. Wen, The sequence of return words of the Fibonacci sequence *Theoret. Comput. Sci.* **593** (2015), 106--116.

- C. Kimberling, K. B. Stolarsky, Slow Beatty sequences, devious convergence, and partitional divergence, *Amer. Math. Monthly* **123** (2016), 267–273.
- C. Kimberling, Complementary equations and Wythoff sequences, *J. Integer Seq.* **11** (2008), Art. 08.3.3.
- U. Larsson, N. A. McKay, R. J. Nowakowski, A. A. Siegel, Finding golden nuggets by reduction, Preprint (2015), https://arxiv.org/abs/1510.07155
- D. A. Lind, The quadratic field $Q(\sqrt{5})$ and a certain Diophantine equation, *Fibonacci Quart.* **6** (1968), 86–93.
- A. McD. Mercer, Generalized Beatty sequences, *Int. J. Math. Math. Sci.*, **1** (1978), 525–528.
- J. Lambek, L. Moser, Inverse and complementary sequences of natural numbers, *Amer. Math. Monthly* **61** (1954), 454–458.
- *On-Line Encyclopedia of Integer Sequences*, founded by N. J. A. Sloane, electronically available at http://oeis.org.

- M. E. Paul, Minimal symbolic flows having minimal block growth, *Math. Systems Theory* **8** (1975), 309–315.
- R. Tijdeman, On complementary triples of Sturmian bisequences, *Indag. Math.* **7** (1996), 419–424.
- R. Tijdeman, Exact covers of balanced sequences and Fraenkel's conjecture. in *Algebraic number theory and Diophantine analysis (Graz, 1998)*, de Gruyter, Berlin, 2000, pp. 467–483.
- J. V. Uspensky, On a problem arising out of the theory of a certain game, *Amer. Math. Monthly* **34** (1927), 516–521.
- Z.-X. Wen, Z.-Y. Wen, Some properties of the singular words of the Fibonacci word, European J. Combin. 15 (1994) 587598.