On Hadamard $\mathbb{Q}$-Series and Rotating $\mathbb{Q}$-Automata

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Outline

1. Automata
   - $\mathbb{Q}$-Automata
   - Rotating automata
   - Validity

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   - Rational series
   - Hadamard series

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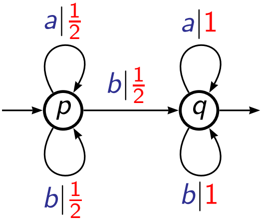
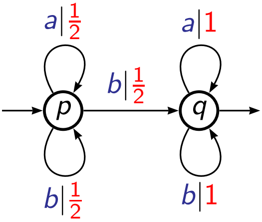
$\mathbb{Q}$-automata



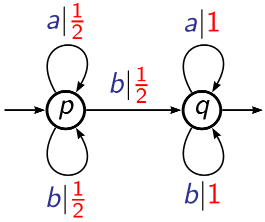
$\mathbb{Q}$-automata



Weight of a word

Sum of the weights of the accepting runs.

$\mathbb{Q}$-automata



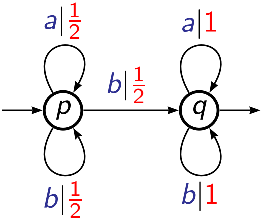
Weight of a word

Sum of the weights of the accepting runs.

Weight of a run

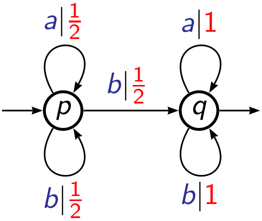
Product of the weights of the transitions.

$\mathbb{Q}$-automata



b a b b Weight of the run

$\mathbb{Q}$-automata

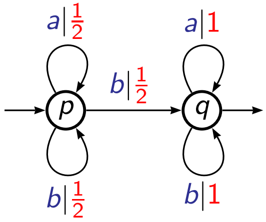


b a b b

$$\rightarrow p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} q \rightarrow$$

Weight of the run

$$\frac{1}{16}$$

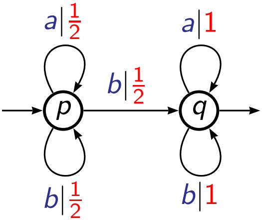


b	a	b	b	Weight of the run

$$\rightarrow p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} q \rightarrow \qquad \frac{1}{16}$$

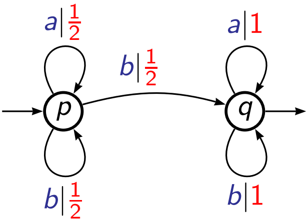
$$\rightarrow p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} p \xrightarrow{\frac{1}{2}} q \xrightarrow{1} q \rightarrow \qquad + \quad \frac{1}{8}$$

$\mathbb{Q}$-automata



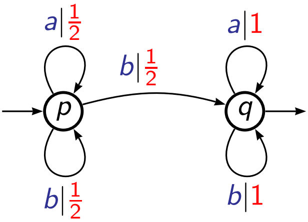
	b		a		b		b		Weight of the run
$\to$p	$\xrightarrow{\frac{1}{2}}$	p	$\xrightarrow{\frac{1}{2}}$	p	$\xrightarrow{\frac{1}{2}}$	p	$\xrightarrow{\frac{1}{2}}$	q$\to$	$\frac{1}{16}$
$\to$p	$\xrightarrow{\frac{1}{2}}$	p	$\xrightarrow{\frac{1}{2}}$	p	$\xrightarrow{\frac{1}{2}}$	q	$\xrightarrow{1}$	q$\to$	$+ \ \frac{1}{8}$
$\to$p	$\xrightarrow{\frac{1}{2}}$	q	$\xrightarrow{1}$	q	$\xrightarrow{1}$	q	$\xrightarrow{1}$	q$\to$	$+ \ \frac{1}{2}$
									$= \ [0.1011]_2$

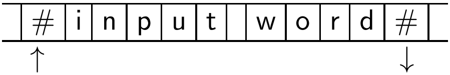
Rotating $\mathbb{Q}$-Automata



$$s : \quad w \mapsto [0.w]_2$$

Rotating ℚ-Automata





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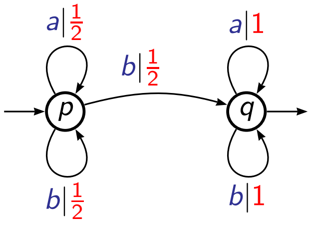
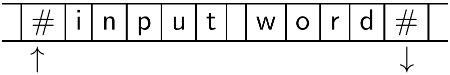
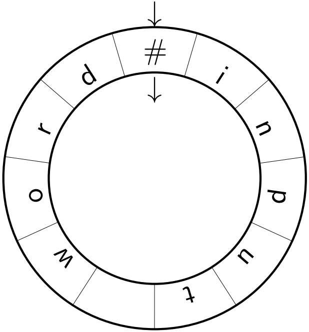
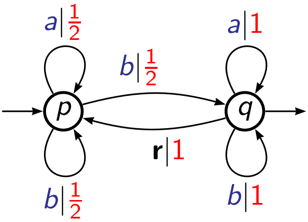
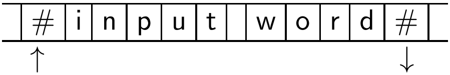


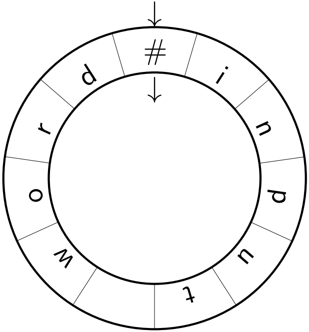




Rotating $\mathbb{Q}$-Automata







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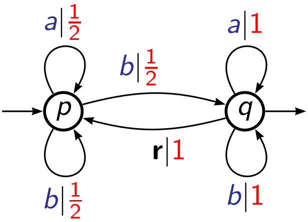
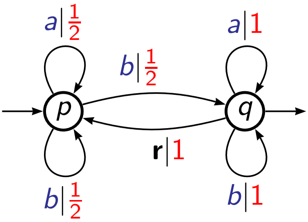


$$\langle \mathcal{A}_{rot}, w \rangle = \sum_{i=0}^{\infty} \langle \mathcal{A}_{1w}, w(\mathbf{r}w)^i \rangle$$

Rotating $\mathbb{Q}$-Automata



$$\langle \mathcal{A}_{rot}, w \rangle = \sum_{i=0}^{\infty} \langle \mathcal{A}_{1w}, w(\mathbf{r}w)^i \rangle$$

$$t : \; w \mapsto ([0.w]_2)^+ = \frac{[0.w]_2}{1 - [0.w]_2}$$

Proposition

Rotating $\mathbb{Q}$-automata are more powerful than one-way $\mathbb{Q}$-automata.

Validity

There may be infinitely many accepting runs on some words.
The sum of their weights may be undefined.
We say in this case that the automaton is not valid.

Proposition

The validity of rotating $\mathbb{Q}$-automata is undecidable.

$\mathbb{Q}$-Series

Definition

$$s : \begin{array}{rcl} A^* & \to & \mathbb{Q} \\ w & \mapsto & \langle s, w\rangle \end{array}, \text{ noted } s = \sum_{w \in A^*} \langle s, w\rangle w.$$

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Example:

$$s_1 = \frac{1}{2}b + \frac{1}{4}ab + \frac{1}{2}ba + \frac{3}{4}bb + \frac{1}{8}aab + \frac{1}{4}aba + \ldots$$

Weighted extension of languages.

Operations on series

		Series	Languages
Sum	$\langle s+t, w\rangle =$	$\langle s, w\rangle + \langle t, w\rangle$	$L_1 \cup L_2$
Cauchy product	$\langle s.t, w\rangle =$	$\sum_{uv=w} \langle s, u\rangle.\langle t, v\rangle$	$L_1.L_2$
Kleene star	$s^* =$	$\sum_{n=0}^{\infty} s^n$	L_1^*
Hadamard product	$\langle s \odot t, w\rangle =$	$\langle s, w\rangle.\langle t, w\rangle$	$L_1 \cap L_2$

The Kleene star is defined iff $\langle s, \varepsilon\rangle^*$ is defined.

Rational series

Rational series $=<\text{Poly}>_{+,.,*}$. $< Rat >_{\odot} = Rat$

Extension of regular languages.

Theorem (Schützenberger 61)

Let s be a series. The following propositions are equivalent:

- s is a $\mathbb{Q}$-rational series.
- s is the behaviour of a $\mathbb{Q}$-automaton;

Inverse

If s is rational and invertible (for the Cauchy product), s^{-1} is rational.

This implication does not hold for the inverse of the Hadamard product, noted $\odot\frac{A^*}{s}$. (A^* is neutral for the Hadamard product)

The Hadamard inverse can also be expressed with $\circledast$, the iteration of the Hadamard product.

Hadamard series

Hadamard series

Hadamard series are the closure of rational series by sum, Hadamard product and Hadamard inverse.

Every Hadamard series is of the form $\odot\frac{s}{t}$, with s and t two rational series.

Validity issues

s is Cauchy-invertible iff $\langle s, \varepsilon \rangle \neq 0$
This is decidable.

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s is Hadamard-invertible iff $\forall w, \langle t, w \rangle \neq 0$.

Proposition

Hadamard invertibility is undecidable.

The description $\odot\frac{s}{t}$ for a Hadamard series may be not defined.

A Schützenberger-like theorem

Theorem 1

Let s be a series. The following propositions are equivalent:

- s is a $\mathbb{Q}$-Hadamard series;
- s is the behaviour of a rotating $\mathbb{Q}$-automaton;

The conversions from one description to another one are effective (if the representations are correct).

A Schützenberger-like theorem

Theorem 1

Let s be a series. The following propositions are equivalent:

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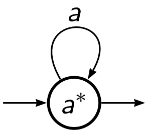
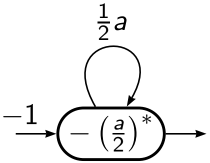
Equivalence of rotating $\mathbb{Q}$-automata is decidable

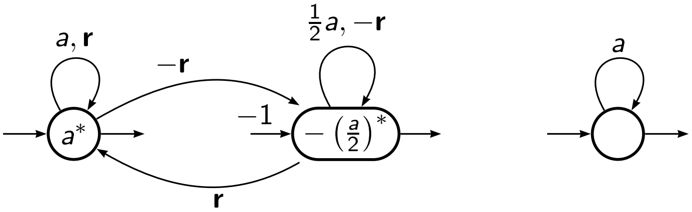
$\odot\frac{s}{t} = \odot\frac{s'}{t'} \Leftrightarrow s \odot t' = s' \odot t$, with s, t, s', t' rational.

Construction





$$\mathcal{A}: \; a^* - \left(\frac{a}{2}\right)^*$$



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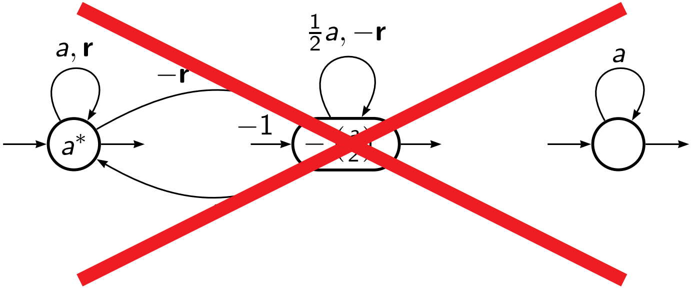

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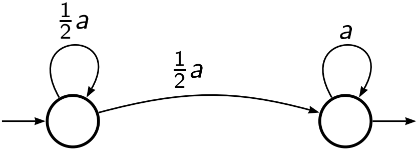
Correct construction

There is nonetheless a construction that yields a valid automaton.



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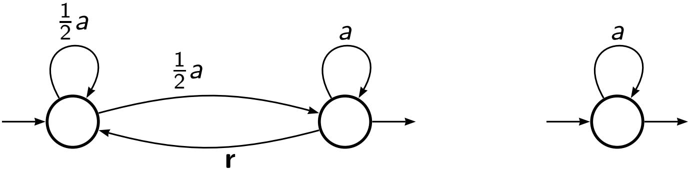

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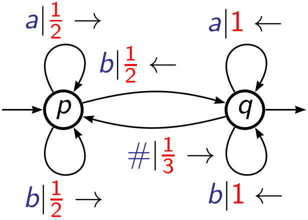
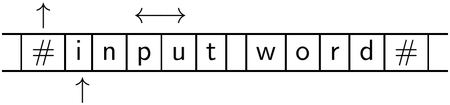
Two-way automata





Comparison between models

Finite semirings	One-way	$=$	Rotating	$=$	Two-way
$\mathbb{Q}$-automata (or $\mathbb{R}$ or $\mathbb{C}$)	One-way	$\subsetneq$	Rotating	?	Two-way
Transducers	One-way	$\subsetneq$	Rotating	$\subsetneq$	Two-way

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Two-way to rotating

Algebraic proof (computation of a determinant of the star of a matrix).
No idea for a more combinatorial proof, that maybe could be true outside $\mathbb{C}$.

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Thank you for your attention