On the group of a rational maximal bifix code

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joint work with Jorge Almeida, Revekka Kyriakoglou & Dominique Perrin

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Preview

- Relativization of maximal bifix codes:

  filtration

$$X = Z \cap F$$

  of a maximal bifix code Z by a (uniformly) recurrent set F

- Relativization of descriptors of maximal bifix codes:
  - $d(Z) \rightsquigarrow d_F(X)$
  - $G(Z) \rightsquigarrow G_F(X)$

- We give necessary and sufficient conditions for

$$G(Z) \cong G_F(X)$$

- Methodological novelty: the use of free profinite monoids

The syntactic monoid $M(L)$ of a language L is the transition monoid of the minimal automaton of L

We denote the map $A^* \to M(L)$ by η_L

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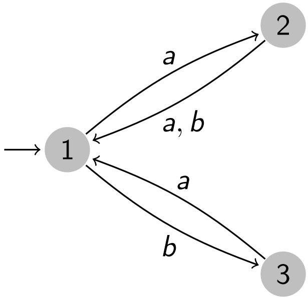
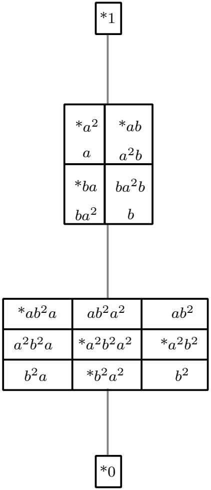
e.g., $\quad L = \{aa, ab, ba\}^*$



$M(L)$:



Green's relations

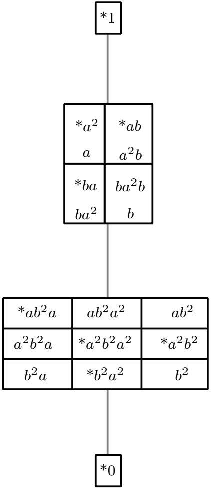
1. $u \mathcal{R} v \Leftrightarrow uM = vM$
2. $u \mathcal{L} v \Leftrightarrow Mu = Mv$
3. $u \mathcal{J} v \Leftrightarrow MuM = MvM$
4. $u \leq_{\mathcal{J}} v \Leftrightarrow MuM \subseteq MvM$
5. $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
6. In a finite monoid,

$$\mathcal{J} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \vee \mathcal{L}$$

   All monoids we consider have this property.

7. For such monoids, if a $\mathcal{J}$-class contains an idempotent, then its $\mathcal{H}$-classes containing idempotents are isomorphic subgroups.

   The abstract semigroup thus defined is the Schützenberger group of the $\mathcal{J}$-class.



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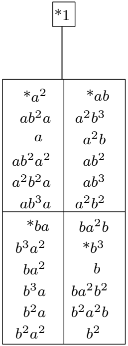

Maximal bifix codes

A bifix code X of A^* is maximal if

$$X \subseteq Y \text{ and } Y \text{ is bifix} \implies X = Y$$

Motivation

- Perhaps the most studied and "tractable" class of codes.
- Important case: if $M(X^*)$ is a group, then X is a maximal bifix code, called a group code.

Filtration by recurrent sets

A bifix code X is F-maximal if $X \subseteq F$ and

$$X \subseteq Y \subseteq F \text{ and } Y \text{ is bifix} \implies X = Y$$

Theorem (Berstel & De Felice & Perrin & Reutenauer & Rindone; 2012)

If Z is maximal bifix, then $X = Z \cap F$ is...

- *... F-maximal bifix if F is recurrent*
- *... finite if F is uniformly recurrent*

Degree

If X is a rational maximal bifix code, then the degree of X is the rank $d(X)$ of the minimum ideal $J(X)$ of $M(X^*)$.

Theorem (The five authors of the 2012 paper + Dolce & Leroy; 2015)

If Z is a maximal bifix code and F is a tree set, then $X = Z \cap F$ is a basis of a subgroup of index $d(Z)$ of the free group $FG(A)$.

F-degree

Let X be a rational maximal bifix code.

The F-minimum $\mathcal{J}$-class of $M(X^*)$, denoted $J_F(X)$, is the $\mathcal{J}$-class of $M(X^*)$ that is $\mathcal{J}$-minimum among the $\mathcal{J}$-classes intersecting $\eta_{X^*}(F)$.

The F-degree of X is the rank $d_F(X)$ of $J_F(X)$.

F-groups

The group of Z, denoted $G(Z)$, is the Schützenberger group of $J(Z)$.

The F-group of X, denoted $G_F(X)$, is the Schützenberger group of $J_F(X)$.

- $G(Z)$ is a permutation group of degree $d(Z)$
- $G_F(X)$ is a permutation group of degree $d_F(X)$

We want to relate $G(Z)$ with $G_F(X)$...

Theorem (Berstel & De Felice & Perrin & Reutenauer & Rindone; 2012)

If Z is a group code and F is Sturmian, then $G(Z) \simeq G_F(X)$ and $d(Z) = d_F(X)$.

(the theorem does not hold if we just replace "Sturmian" by "uniformly recurrent", or "group code" by "maximal bifix code")

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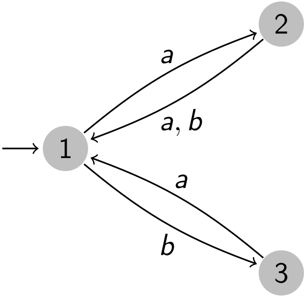
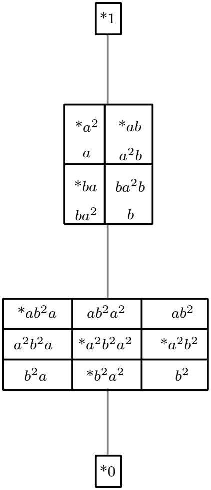
- $Z = \{aa, ab, ba, bb\}$
- $F =$ "Fibonacci set"

Minimum automaton of X^*, where $X = Z \cap F$:



$$G(Z) \simeq G_F(X) \simeq \mathbb{Z}/2\mathbb{Z}$$

$M(X^*)$:



The profinite completion of A^*

If x, y are distinct elements of A^*, then there is a finite quotient $M = A^*/\sim$ separating x and y (i.e. $x \not\sim y$).

Let $r(x, y)$ be the smallest possible cardinal for M.

$$d(x, y) = 2^{-r(x,y)}$$

For this metric, let $\widehat{A^*}$ be the metric completion of A^*.

$\widehat{A^*}$ is a topological monoid, with a profinite topology.

More: $\widehat{A^*}$ is the free profinite monoid generated by A.

The group $G(F)$

If F is recurrent, then there is a $\mathcal{J}$-minimum $\mathcal{J}$-class contained in $\overline{F}$, denoted $J(F)$.

The group $G(F)$ is the (topological!) Schützenberger group of $J(F)$.

Tree and connected case

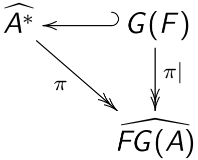
Theorem (Almeida & Costa; 2017)

If F is a tree set, then $G(F)$ is a free profinite group of rank $|A|$.

(the proof uses the "Return Theorem")

More precisely:

if F is a tree set, then $\pi|\colon G(F) \to \widehat{FG(A)}$ is an isomorphism:



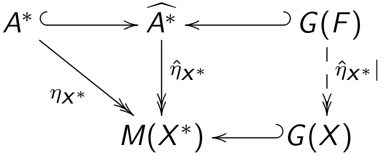
More generally, $\pi|\colon G(F) \to \widehat{FG(A)}$ is onto if F is connected.

F-charged codes

We say that a code X is:

1. F-charged if

$$\hat{\eta}_{X^*}(G(F)) = G(X)$$



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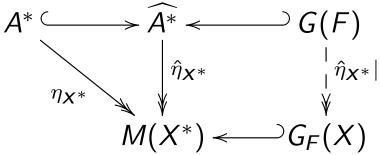

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Necessary and sufficient conditions for $G(Z) \simeq G_F(Z \cap F)$

Theorem (Almeida & Costa & Kyriakoglou & Perrin; 2018)

Let:

- *F recurrent*
- *Z rational maximal bifix code*

Suppose also that $X = Z \cap F$ is rational.

The following conditions are equivalent:

- *Z is F-charged*
- *$d_F(X) = d(Z)$, $G_F(X) \simeq G(Z)$ and X is weakly F-charged*

Additionally: if F is uniformly recurrent, then the equality $d_F(X) = d(Z)$ is redundant in the second condition.

Necessary and sufficient conditions for $G(Z) \simeq G_F(Z \cap F)$

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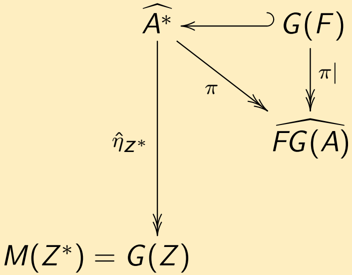
Additionally: if F *is uniformly recurrent, then the equality* $d_F(X) = d(Z)$ *is redundant in the second condition.*

Group codes are "connected"-charged

Fact

If Z is a group code and F is connected, then Z is F-charged.

Proof.



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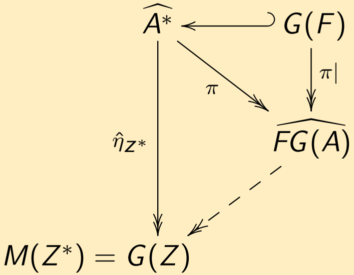


□

Primitive substitutions

When $F = F_\varphi$ is described by a primitive substitution φ, we have an algorithm to decide if X is (weakly) F-charged: this is done via a profinite presentation for $G(F_\varphi)$, obtained by Almeida & Costa (2013).

Example

- F_τ: the Prouhet-Thue-Morse set, where

$$\tau : a \mapsto ab, b \mapsto ba$$

- Z: group code over $\{a, b\}$ generating the stabilizer of 1 via

$$a \mapsto (123), \quad b \mapsto (345)$$

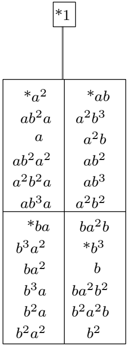
- $G(Z) = A_5$
- Z is F_τ-charged

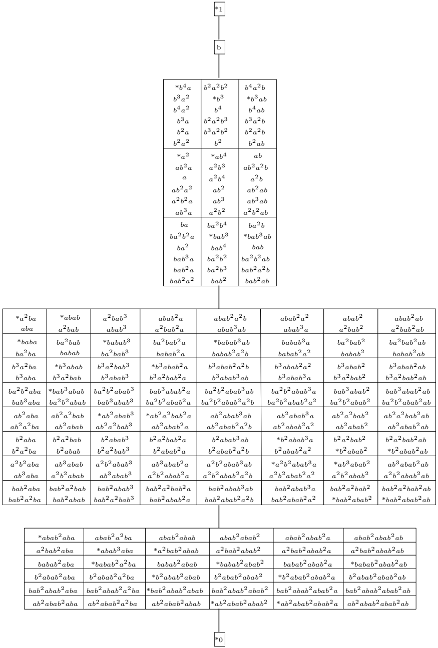
- $Z = \{aa, ab, ba\} \cup b^2(a^+b)^*b$
- $F = A^* \setminus A^*ab(b^2)^*aA^*$

Z is maximal bifix, but not a group code

F is recurrent, but not uniformly

Z is F-charged and $G(Z) \simeq G_F(X) \simeq S_3$





- $Z = \{aa, ab, ba\} \cup b^2(a^+b)^*b$
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Z is maximal bifix, but not a group code

F is Sturmian

Z is not F-charged and $G(Z) \not\simeq G_F(X)$

$*1$

$*a^2$ ab^2a a ab^2a^2 a^2b^2a ab^3a	$*ab$ a^2b^3 a^2b ab^2 ab^3 a^2b^2
$*ba$ b^3a^2 ba^2 b^3a b^2a b^2a^2	ba^2b $*b^3$ b ba^2b^2 b^2a^2b b^2

$*1$

$*a^2$ a	$*ab$ a^2b
$*ba$ ba^2	ba^2b b

$*ab^2a$	ab^2a^2	ab^2
a^2b^2a	$*a^2b^2a^2$	$*a^2b^2$
b^2a	$*b^2a^2$	b^2

$*0$

Conclusion: profinite is good!

We gave the first relevant "external" applications of the profinite Schützenberger group $G(F)$ of a uniformly recurrent set.

- the statement

$$G(Z) \cong G_F(Z \cap F) \text{ if } F \text{ is connected and } Z \text{ is group code}$$

uses no "profinite jargon"

- the definition of "F-charged" gives a comprehensive framework to improve the latter
- the profinite group $G(F)$ serves as a sort of universal cover for F-groups
- "profinite stuff" facilitates synthetic statements
- advantages of the profinite monoid for proofs:
  - more "conceptual" proofs (diagram style)
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