On the group of a rational maximal bifix code

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joint work with Jorge Almeida, Revekka Kyriakoglou & Dominique Perrin

17e journées montoises d’informatique théorique — 2018
• Relativization of maximal bifix codes:

\[ X = Z \cap F \]

of a maximal bifix code \( Z \) by a (uniformly) recurrent set \( F \)

• Relativization of descriptors of maximal bifix codes:
  • \( d(Z) \cong d_F(X) \)
  • \( G(Z) \cong G_F(X) \)

• We give necessary and sufficient conditions for

\[ G(Z) \cong G_F(X) \]

• Methodological novelty: the use of free profinite monoids
The syntactic monoid $M(L)$ of a language $L$ is the transition monoid of the minimal automaton of $L$

We denote the map $A^* \to M(L)$ by $\eta_L$

e.g., $L = \{aa, ab, ba\}^*$
Green’s relations

1. \( u \mathcal{R} v \iff uM = vM \)
2. \( u \mathcal{L} v \iff Mu = Mv \)
3. \( u \mathcal{J} v \iff MuM = MvM \)
4. \( u \leq \mathcal{J} v \iff MuM \subseteq MvM \)
5. \( \mathcal{H} = \mathcal{R} \cap \mathcal{L} \)
6. In a finite monoid,

\[ \mathcal{J} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \lor \mathcal{L} \]

All monoids we consider have this property.

For such monoids, if a \( \mathcal{J} \)-class contains an idempotent, then its \( \mathcal{H} \)-classes containing idempotents are isomorphic subgroups.

The abstract semigroup thus defined is the Schützenberger group of the \( \mathcal{J} \)-class.
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Maximal bifix codes

A bifix code $X$ of $A^*$ is maximal if

$$X \subseteq Y \text{ and } Y \text{ is bifix} \implies X = Y$$

Motivation

- Perhaps the most studied and “tractable” class of codes.
- Important case: if $M(X^*)$ is a group, then $X$ is a maximal bifix code, called a group code.
A bifix code $X$ is $F$-maximal if $X \subseteq F$ and

$$X \subseteq Y \subseteq F \text{ and } Y \text{ is bifix} \implies X = Y$$

**Theorem** (Berstel & De Felice & Perrin & Reutenauer & Rindone; 2012)

*If $Z$ is maximal bifix, then $X = Z \cap F$ is...*

- ... **$F$-maximal bifix** if $F$ is recurrent
- ... **finite** if $F$ is uniformly recurrent
If $X$ is a rational maximal bifix code, then the degree of $X$ is the rank $d(X)$ of the minimum ideal $J(X)$ of $M(X^*)$.

**Theorem (The five authors of the 2012 paper + Dolce & Leroy; 2015)**

If $Z$ is a maximal bifix code and $F$ is a tree set, then $X = Z \cap F$ is a basis of a subgroup of index $d(Z)$ of the free group $FG(A)$. 
Let $X$ be a rational maximal bifix code.

The $F$-minimum $\mathcal{J}$-class of $M(X^*)$, denoted $J_F(X)$, is the $\mathcal{J}$-class of $M(X^*)$ that is $\mathcal{J}$-minimum among the $\mathcal{J}$-classes intersecting $\eta_{X^*}(F)$.

The $F$-degree of $X$ is the rank $d_F(X)$ of $J_F(X)$.
The group of $Z$, denoted $G(Z)$, is the Schützenberger group of $J(Z)$.

The $F$-group of $X$, denoted $G_F(X)$, is the Schützenberger group of $J_F(X)$.

- $G(Z)$ is a permutation group of degree $d(Z)$
- $G_F(X)$ is a permutation group of degree $d_F(X)$

We want to relate $G(Z)$ with $G_F(X)$...

**Theorem (Berstel & De Felice & Perrin & Reutenauer & Rindone; 2012)**

If $Z$ is a group code and $F$ is Sturmian, then $G(Z) \simeq G_F(X)$ and $d(Z) = d_F(X)$.

*(the theorem does not hold if we just replace “Sturmian” by “uniformly recurrent”, or “group code” by “maximal bifix code”)*
**F-groups**

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- $Z = \{aa, ab, ba, bb\}$
- $F = \text{“Fibonacci set”}$

Minimum automaton of $X^*$, where $X = Z \cap F$:

$G(Z) \simeq G_F(X) \simeq \mathbb{Z}/2\mathbb{Z}$
The profinite completion of $A^*$

If $x, y$ are distinct elements of $A^*$, then there is a finite quotient $M = A^*/\sim$ separating $x$ and $y$ (i.e. $x \not\sim y$).

Let $r(x, y)$ be the smallest possible cardinal for $M$.

$$d(x, y) = 2^{-r(x, y)}$$

For this metric, let $\widehat{A}^*$ be the metric completion of $A^*$.

$\widehat{A}^*$ is a topological monoid, with a profinite topology.

More: $\widehat{A}^*$ is the free profinite monoid generated by $A$. 
The group $G(F)$

If $F$ is recurrent, then there is a $\mathcal{J}$-minimum $\mathcal{J}$-class contained in $\overline{F}$, denoted $J(F)$.

The group $G(F)$ is the (topological!) Schützenberger group of $J(F)$.
Tree and connected case

**Theorem (Almeida & Costa; 2017)**

*If $F$ is a tree set, then $G(F)$ is a free profinite group of rank $|A|$.*

*(the proof uses the “Return Theorem”)*

More precisely:

if $F$ is a tree set, then $\pi| : G(F) \to \widehat{FG}(A)$ is an isomorphism:

More generally, $\pi| : G(F) \to \widehat{FG}(A)$ is onto if $F$ is connected.
We say that a code $X$ is:

1. **$F$-charged** if
   \[ \hat{\eta}_{X^*}(G(F)) = G(X) \]

2. **weakly $F$-charged** if
   \[ \hat{\eta}_{X^*}(G(F)) = G_F(X) \]
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Theorem (Almeida & Costa & Kyriakoglou & Perrin; 2018)

Let:
- $F$ recurrent
- $Z$ rational maximal bifix code

Suppose also that $X = Z \cap F$ is rational.

The following conditions are equivalent:
- $Z$ is $F$-charged
- $d_F(X) = d(Z)$, $G_F(X) \cong G(Z)$ and $X$ is weakly $F$-charged

Additionally: if $F$ is uniformly recurrent, then the equality $d_F(X) = d(Z)$ is redundant in the second condition.
Necessary and sufficient conditions for $G(Z) \simeq G_F(Z \cap F)$

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- $Z$ rational maximal bifix code

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Group codes are “connected”-charged

Fact
If $Z$ is a group code and $F$ is connected, then $Z$ is $F$-charged.

Proof.

\[ M(Z^*) = G(Z) \]
Group codes are “connected”-charged

Fact
If $Z$ is a group code and $F$ is connected, then $Z$ is $F$-charged.

Proof.

\[
\begin{align*}
\hat{A}^* &\quad G(F) \\
\hat{\eta}_Z &\quad \pi \quad \pi| \\
M(Z^*) &\quad FG(A) \\
M(Z^*) = G(Z) &
\end{align*}
\]
Primitive substitutions

When $F = F_\varphi$ is described by a primitive substitution $\varphi$, we have an algorithm to decide if $X$ is (weakly) $F$-charged: this is done via a profinite presentation for $G(F_\varphi)$, obtained by Almeida & Costa (2013).

Example

- $F_\tau$: the Prouhet-Thue-Morse set, where
  \[ \tau : a \mapsto ab, \quad b \mapsto ba \]

- $Z$: group code over $\{a, b\}$ generating the stabilizer of 1 via
  \[ a \mapsto (123), \quad b \mapsto (345) \]

- $G(Z) = A_5$

- $Z$ is $F_\tau$-charged
\[ Z = \{aa, ab, ba\} \cup b^2(a^+b)*b \]

\[ F = A^* \setminus A^*ab(b^2)*aA^* \]

*Z* is maximal bifix, but not a group code

*F* is recurrent, but not uniformly

*Z* is *F*-charged and \( G(Z) \sim G_F(X) \sim S_3 \)
\[ Z = \{aa, ab, ba\} \cup b^2(a+b)^*b \]

- \( F = \text{“Fibonacci set”} \)

\( Z \) is maximal bifix, but not a group code
\( F \) is Sturmian

\( Z \) is not \( F \)-charged and \( G(Z) \neq G_F(X) \)
We gave the first relevant “external” applications of the profinite Schützenberger group $G(F)$ of a uniformly recurrent set.

- the statement
  
  $$G(Z) \cong G_F(Z \cap F)$$
  
  if $F$ is connected and $Z$ is group code

  uses no “profinite jargon“

- the definition of “$F$-charged” gives a comprehensive framework to improve the latter

- the profinite group $G(F)$ serves as a sort of universal cover for $F$-groups

- “profinite stuff” facilitates synthetic statements

- advantages of the profinite monoid for proofs:
  - more “conceptual” proofs (diagram style)
  - enhanced combinatorics (“pseudowords” can be idempotent!)
Conclusion: profinite is good!

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