Number of valid decompositions of Fibonacci prefixes

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A positive integer $n = F_{m_k} + F_{m_{k-1}} + \cdots + F_{m_0}$, where

$m_k > m_{k-1} > \cdots > m_0 \geq 2, \quad F_0 = 0, \quad F_1 = 1, \quad F_2 = 1,$
and $F_{m+2} = F_{m+1} + F_m \quad$ for all $m \geq 0$.

If for all $i \geq 0, \quad m_{i+1} - m_i \geq 2$, we have a canonic representation of n which is unique.

Actually this system was invented by a dutch mathematician, Lekkerkerker, in 1952.

Example

$16 = 13 + 3 = F_7 + F_4 = [100100]_F$

Legal representations

There are multiple representations for the same integer obtained by using

$$\cdots 100 \cdots \longleftrightarrow \cdots 011 \cdots$$

Example

$16 = 13 + 3 = 8 + 5 + 3 = 8 + 5 + 2 + 1 = 13 + 2 + 1$
$16 = [100100]_F = [11100]_F = [11011]_F = [100011]_F$

Valid representations

We allow more freedom to the previous system by using

$$\cdots k0l \cdots \longleftrightarrow \cdots (k-1)\, 1 (l+1) \cdots$$

for all $k > 0$, $l \geq 0$.

We go from $kF_{m+1} + lF_{m-1}$ to $(k-1)F_{m+1} + F_m + (l+1)F_{m-1}$.

Example

$16 = [100100]_F = [11100]_F = [11011]_F = [100011]_F$ are legal representations.
$16 = [10121]_F, [1221]_F, [20000]_F$ are rerpresentations obtained by the previous transformation.

There are 7 valid representations of 16. We note $V(16) = 7$.

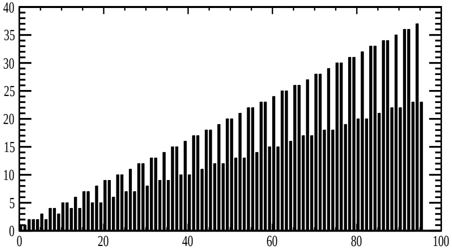


Figure: First 100 values of $V(n)$

Notations and some usual notions

- The lenght of a finite word u is denoted by $|u|$.
- u^k is the concatenation $\underbrace{u \cdots u}_{k}$.
- The i'th symbol of a finite or infinite word u is denoted by $u[i]$, so that $u = u[1]u[2] \cdots$.
- A factor $u[i+1]u[i+2] \cdots u[j]$ of a finite or infinite word u is denoted by $u(i..j]$.
- Then, for $j \geq 0$, the word $u(0..j]$ is the prefix of u of length j.

The Fibonacci sequence

- We define Fibonacci words with the binary alphabet $\{a, b\}$ as follow: $s_{-1} = b, \quad s_0 = a, \quad s_{n+1} = s_n s_{n-1} \quad$ for all $n \geq 0$.
- $s_1 = ab, \quad s_2 = aba, \quad s_3 = abaab, \quad s_4 = abaababa$, and so on.
- The length of s_n is the Fibonacci number F_{n+2}.
- The infinite Fibonacci word is

$$\mathbf{s} = \lim_{n\to\infty} s_n = abaababaabaababaababa\cdots$$

- We note $\mathbf{s}[1] = a$.

- In the *Fibonacci numeration system*, a non-negative integer $N < F_{n+3}$ is represented as

$$N = \sum_{0 \leq i \leq n} k_i F_{i+2}$$

where $k_i \in \{0, 1\}$ for $i \geq 0$.

This is the same system we studied but written differently.

- We have a unique representation of N if the following condition holds:

$$\text{for } i \geq 1, \text{ if } k_i = 1, \text{ then } k_{i-1} = 0$$

- $N = \sum_{0 \leq i \leq n} k_i F_{i+2}$ is represented by $N = [k_n \cdots k_0]_F$.

Lemma 1

For all $k_0, \ldots, k_n$ such that $k_i \in \{0, 1\}$, the word $s_n^{k_n} s_{n-1}^{k_{n-1}} \cdots s_0^{k_0}$ is a prefix of the Fibonacci word $\mathbf{s}$.

Then, a representation of $N = [k_n \cdots k_0]_F$ is *valid* if $k_i \geq 0$ for all i and $\mathbf{s}(0..N] = s_n^{k_n} s_{n-1}^{k_{n-1}} \cdots s_0^{k_0}$.

Example

$\mathbf{s}(0..14] = (abaab)(aba)(aba)(aba)$ is a factorization of 14.
Then, the representation $14 = [1300]_F$ is valid.

The number of valid representations of an integer is exactly the number of factorizations of the corresponding prefix of the Fibonacci word.

Classic properties of the Fibonacci word

- The Fibonacci word $\mathbf{s} = abaababa\cdots$ is the fixed point of the Fibonacci morphism

$$\mu : a \to ab, b \to a$$

- For each $n \geq 1$, we have $s_n = \mu(s_{n-1})$. Then Lemma 1 implies that

$$\mu(\mathbf{s}(0..N]) = \mathbf{s}(0..[k_n \cdots k_0 0]_F]$$

- For all n, we have $\mathbf{s}[n] = \begin{cases} a, & \text{if } \{n/\varphi^2\} < 1 - 1/\varphi^2; \\ b, & \text{otherwise.} \end{cases}$

  Where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Results

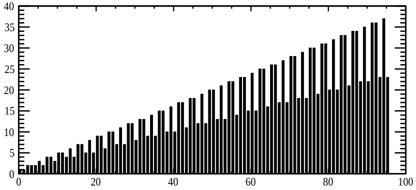
Proposition 1

If $\mathbf{s}[n] = a$, all valid representations of n end with an even number of 0s. If $\mathbf{s}[n] = b$, all of them end with an odd number of 0s.

Theorem 1

If $\mathbf{s}[n] = a$, then $V(n) = \lceil n/\varphi^2 \rceil$,or, equivalently, $V(n)$ is equal to the number of occurrences of b in $\mathbf{s}(0..n]$, plus one.

If $\mathbf{s}[n] = b$, then $V(n) = \lceil n/\varphi^3 \rceil$, or, equivalently, $V(n)$ is equal to the number of occurrences of aa in $\mathbf{s}(0..n]$, plus one.



Proposition 2

(a) $V([r0]_F) \geq V([r]_F)$ for all $r \in \{0,1\}^*$.

(b) If $r = r'10^{2k}$ for some $k \geq 0$, then $V([r0]_F) = V([r]_F)$.

Proposition 3

For all $z \in \{0,1\}^*$ and all $k \geq 1$, we have

$$V([z10^{2k}]_F) = V([z10^{2k-2}]_F) + V([z(01)^k]_F)$$

Proposition 4

For all $z \in \{0,1\}^*$ and all $k \geq 1$, we have

$$V([z10^k1]_F) = \begin{cases} V([z10^{k+1}]_F), & \text{if } k \text{ is odd;} \\ V([z10^k]_F) + V([z(01)^{k/2}]_F), & \text{if } k \text{ is even.} \end{cases}$$

Corollary 1

For all $k \geq 1$, we have $\quad V(F_{2k+2} - 2) = F_{2k}$

$$\text{and} \quad V(F_{2k+1} - 1) = V(F_{2k+1} - 2) = F_{2k-1}$$

Corollary 2

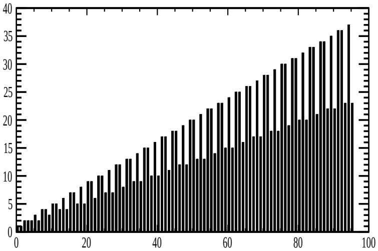
For all $k \geq 1$, we have

$$V(F_{2k}) = V(F_{2k+1}) = F_{2k-2} + 1.$$

Proposition 5

Let $n = [z]_F$ and $n' = [z0]_F$ be such that $\mathbf{s}[n] = a$.

Then $\lceil n/\varphi^2 \rceil = \lceil n'/\varphi^3 \rceil$.



The theorem ensures that the sequence $(V(n))$ grows as shown on the graph. The two visible straight lines correspond to the symbols of the Fibonacci word equal to a (the upper line) or b (the lower line).

Fibonacci-regular representation

Jeffrey Shallit added this part to show that the sequence $(V(n))$ is Fibonacci-regular.

A sequence $(s(n))_{n\geq 0}$ is said to be *Fibonacci-regular* if there exist an integer k, a row vector v of dimension k, a column vector w of dimension k, and a $k\times k$ matrix-valued morphism ρ such that for $z\in\{0,1\}^*$,

$$s([z]_F) = v\rho(z)w$$

The triple (v,ρ,w) is called a *linear representation*.

For all $x \in \{0,1\}^*$,

$$\begin{aligned} V(x01) &= -V(x) + V(x0) + V(x00) \\ V(x10) &= V(x1) \\ V(x0100) &= -V(x) + 2V(x00) + V(x000) \\ V(x1000) &= V(x100) \\ V(x010000) &= -V(x) - V(x0) + 2V(x00) + 3V(x000) + V(x0000) \\ V(x00000) &= V(x) - V(x0) - 3V(x00) + 3V(x000) + V(x0000) \end{aligned}$$

We can demonstrate these relations thanks to the previous propositions and they are used to proove that $(V(n))$ is Fibonacci-regular.