Unary Patterns of Size Four with Morphic Permutations

Kamellia Reshadi

Institut für Informatik, Christian-Albrechts-Universität zu Kiel.
kre@informatik.uni-kiel.de

Abstract. We investigate the avoidability of unary patterns of size of four with morphic permutations. More precisely, we show how to identify precisely, given the positive integers i, j, k, the alphabets over which a pattern $x\pi^i(x)\pi^j(x)\pi^k(x)$ is avoidable, where x is a word variable and π is a function variable with values in the set of all morphic permutations of the respective alphabets. This continues the work of [Manea et al., 2015], where a complete characterisation of the avoidability of cubic patterns with permutations was given.

1 Introduction

The avoidability of patterns in infinite words is an old area of interest with a first systematic study going back to Thue. In these initial papers it was shown that there exist a binary infinite morphic word and a ternary infinite morphic word that avoid cubes and squares, respectively. That is, these infinite words do not contain instances of the patterns xxx and xx, respectively.

In this article, we are studying the avoidability of repetitions in a generalised setting. Namely, we are interested in the avoidability of unary patterns with functional dependencies between variables. We are considering patterns like $x\pi^i(x)\pi^j(x)\pi^k(x)$, where x is a word variable while π is function variable, which can be replaced by bijective morphisms only. The instances of such patterns over an alphabet Σ are obtained by replacing x with a concrete word, and π by a morphic permutation of Σ. For example, an instance of the pattern $x\pi(x)x\pi(x)$ over $\Sigma = \{a, b\}$ is the word $uvuv$ such that $|u| = |v|$, and v is the image of u under any permutation on the alphabet. Considering the permutation $a \to b$, and $b \to a$, then $aba|bab|aba|bab$ is an instance of $x\pi(x)x\pi(x)$.

In this setting, we continue the work of [Manea et al., 2015] as follows. In that paper, a complete characterisation of the avoidability of cubic patterns with permutations $x\pi^i(x)\pi^j(x)$ was given. Furthermore, it was shown that there exists a ternary word that avoids all patterns $\pi_{i_1}(x)\dots\pi_{i_r}(x)$ where $r \geq 4$, x a word variable over some alphabet Σ, with $|x| \geq 2$ and $|\Sigma| \geq 3$, and the π_{i_j} function variables that may be replaced by anti-/morphic permutations of Σ. However, this result only holds when the length of x is restricted to be at least 2. Also, it was shown that all patterns $\pi^{i_1}(x)\dots\pi^{i_n}(x)$ with $n \geq 4$ under morphic permutations are avoidable in alphabets of size $2, 3$, and 4, but there exist patterns which are unavoidable in alphabets of size 5. We extend these results by showing how to determine exactly, for a given unary pattern $\mathcal{P}$ of size four with permutations, which are the alphabets in which it is avoidable.

The main result of our paper is that given i, j, k, we show how to compute the value m such that the pattern $x\pi^i(x)\pi^j(x)\pi^k(x)$, with $i, j, k \geq 0$, is unavoidable in alphabets of size at least m and avoidable in alphabets of size $2, 3, 4, \ldots, m-1$. To achieve this, we define a series of parameters that allow us to characterize, for each alphabet, what form the instances of the pattern may have over a certain alphabet. Then, we show that for each pattern there exists an interval (whose left end is 2 and right end is defined based on the respective parameters) such that over each alphabet whose size is in the respective interval, there exists an infinite word that does not contain instances of the given pattern. The structure of the paper is as follows: we first give a series of basic definitions and preliminary results. Then we define the aforementioned parameters, and show how to use them to compute, for a given pattern, the minimum size σ of an alphabet over which the respective pattern is unavoidable. Finally, we show the correctness of the computation done in the previous step: for alphabets with less than σ symbols the pattern is avoidable.

2 Preliminaries

We define $\Sigma_k = \{0, \ldots, k-1\}$ to be an alphabet with k letters; the empty word is denoted by ε. For words u and w, we say that u is a prefix (resp. suffix) of w, if there exists a word v such that $w = uv$ (resp. $w = vu$). If $f : \Sigma_k \to \Sigma_k$ is a permutation, we say that the order of f, denoted $\mathbf{ord}(f)$, is the minimum value $m > 0$ such that f^m is the identity. If $a \in \Sigma_k$ is a letter, the order of a with respect to f, denoted $\mathbf{ord}_f(a)$, is the minimum number m such that $f^m(a) = a$.

In this paper, we consider only unary patterns (i.e., containing only one variable) with morphic permutations, that is, all function variables are unary and are substituted by morphic permutations only.

The infinite Hall word h is defined as $h = \lim_{n\to\infty} \phi_h^n(0)$, for the morphism $\phi_h : \Sigma_3^* \to \Sigma_3^*$ where $\phi_h(0) = 012$, $\phi_h(1) = 02$ and $\phi_h(2) = 1$. The infinite word h avoids the pattern xx (squares).

3 Avoidability of patterns under permutations

In this section we try to identify an upper bound on the size of the alphabets Σ_m in which a patterns $x\pi^i(x)\pi^j(x)\pi^k(x)$, with $i, j, k \geq 0$ is unavoidable, when π is substituted by a morphic permutation.

In the pattern $x\pi^i(x)\pi^j(x)\pi^k(x)$, the factors x, $\pi^i(x)$, π^j, or $\pi^k(x)$ are called x-items in the following. Our analysis is based on the relation between the possible images of the four x-items occurring in a pattern, following the ideas of [**?**]. For instance, we want to check whether in a possible image of our pattern, all four x-items can be mapped to a different word, or whether the second and the last x-items can be mapped to the same word, etc.

To achieve this, we define in Table 1 the parameters α_i, with $1 \leq i \leq 14$.

$\alpha_1 = \inf\{t : t \nmid i, t \nmid j, t \nmid k, t \nmid \lvert i-j \rvert, t \nmid \lvert i-k \rvert, t \nmid \lvert j-k \rvert\}$	0123
$\alpha_2 = \inf\{t : t \mid i, t \nmid j, t \nmid k, t \nmid \lvert j-k \rvert\}$	0012
$\alpha_3 = \inf\{t : t \nmid i, t \mid j, t \nmid k, , t \nmid \lvert i-k \rvert\}$	0102
$\alpha_4 = \inf\{t : t \nmid i, t \nmid j, t \mid \lvert i-k \rvert\}$	0121
$\alpha_5 = \inf\{t : t \nmid i, t \nmid j, t \nmid \lvert i-j \rvert, t \nmid \lvert i-k \rvert, t \mid \lvert j-k \rvert\}$	0122
$\alpha_6 = \inf\{t : t \mid i, t \mid j, t \nmid k\}$	0001
$\alpha_7 = \inf\{t : t \mid i, t \nmid j, t \mid k\}$	0010
$\alpha_8 = \inf\{t : t \nmid i, t \mid j, t \mid k\}$	0100
$\alpha_9 = \inf\{t : t \nmid i, t \mid \lvert i-j \rvert, t \mid \lvert i-k \rvert\}$	0111
$\alpha_{10} = \inf\{t : t \mid i, t \nmid j, t \mid \lvert j-k \rvert\}$	0011
$\alpha_{11} = \inf\{t : t \nmid i, t \mid j, t \mid \lvert i-k \rvert\}$	0101
$\alpha_{12} = \inf\{t : t \nmid i, t \mid k, t \mid \lvert i-j \rvert\}$	0110
$\alpha_{13} = \inf\{t : t \nmid i, t \nmid k, t \mid \lvert i-j \rvert\}$	0112
$\alpha_{14} = \inf\{t : t \nmid i, t \nmid j, t \mid \lvert i-j \rvert\}$	0120

Table 1. Definition of the values α_i, with $1 \leq i \leq 14$.

Now based on combinatorial relations, we define the some collections of sets. The idea behind all these collections is to generate sets of parameters α_is that cannot be avoided and have a minimal cardinality. No matter what will be added to these sets will preserve their unavoidability, while erasing something from them will make them avoidable. To obtain these collections we used a computer program and randomly generated some unavoidable sets of parameters of size five. Using the similarities between the instances modelled by these sets, defined in terms of (gapped) squares and cubes occurring in their digit representation, we developed an algorithm to generate more sets of patterns. Based on these relations we constructed fourteen sets $\mathcal{S}_i$s . We just define one of them as an example.

Let $\mathcal{S}_1$ be the collection of sets (each with five elements) that contain α_1 and:

- one of the α_is whose representation has a prefix or a suffix square, but no gapped cube. That is: α_2 or α_5.
- one of the α_is that has a gapped square, but does not have two gapped squares. These are α_3 or α_4.
- one of the α_is that contain cubes or two squares: α_6 or α_9 or α_{10}.
- one of the α_is that contain gapped cubes: α_7 or α_8.

For example, one possible set from $\mathcal{S}_1$ is $\{\alpha_1, \alpha_2, \alpha_4, \alpha_6, \alpha_7\}$.

Lemma 1 *Let $K' \subset K$ be any subset of size at most 4 of K. There exists an infinite word w such that w does not contain 4-powers and if w contains an instance of the pattern $x\pi^i(x)\pi^j(x)\pi^k(x)$ then it can not be modelled by any tuples of the set of patterns K'.*

Theorem 1. *Let i, j, k be positive integers such that $i \neq j \neq k \neq i$, and consider the pattern $p = x\pi^i(x)\pi^j(x)\pi^k(x)$. Let $\sigma = \min\{\max(S) \mid S \in \cup_{1 \leq i \leq 13} S_i\}$. Then $\sigma \geq 5$ and p is unavoidable in Σ_m, for $m \geq \sigma$.*

Proof. We briefly prove this Theorem. The complete proof is in the main paper. We checked with the aid of a computer, by a straightforward backtracking algorithm, that if $m \geq \max(S)$, for some $S \in \cup_{1 \leq i \leq 13} S_i$, then p is unavoidable in Σ_m. Our computer program tries construct a word as long as possible by always adding a letter to the current word it constructed by backtracking; this letter is chosen in all possible ways from the letters contained in the word already, or it may also be a new letter. $\square$

4 Algorithm to generate avoidable cases

Algorithm 1 Algorithm to generate avoidable cases

1: Let $n = 13$. Using the sets $\mathcal{S}_i$, $(1 \leq i \leq 13)$, generate all sets of α_is of cardinality n, that have no unavoidable sets of patterns as subset; show that they are avoidable;
2: For all n from 12 downto 4, generate all sets of cardinality n that have no unavoidable sets of patterns as subset; these sets should not be subsets of the avoidable sets of α_is of cardinality $n+1$ (to avoid generating repetitive avoidable sets of cases generated in the past step); show that they are avoidable.

Theorem 2. *Given a pattern $p = x\pi^i(x)\pi^j(x)\pi^k(x)$ we can determine effectively the values m, such that the pattern is avoidable in Σ_m.*

Proof. We briefly prove this Theorem. The complete proof is in the main paper. In Theorem 1, we proved that given the pattern$p = x\pi^i(x)\pi^j(x)\pi^k(x)$, for each i, j, and k, we can compute the cardinality of an alphabet over which the pattern is unavoidable. Now to show that this is the minimum cardinality over which the pattern of size four is unavoidable, we proceed as follows. We will show that the subsets and the complements of all the sets $\mathcal{S}_i$,$(1 \leq i \leq 13)$ are avoidable. By complement we mean here all the sets of parameters α_i of which the sets Patterns of Size Four with Morphic Permutations are not subsets of. The reason to define it this way is that if a set of parameters is unavoidable, whatever we add to it remains unavoidable, so this set should not be subset of any avoidable set of patterns. Furthermore, to show that the value $\alpha = \alpha_i$, $1 \leq i \leq 14$ in the set $\mathcal{S}_j, 1 \leq j \leq 13$, is the minimum cardinality of an alphabet over which the pattern of size four is unavoidable, we should prove that α_i is the minimum value such that if we add it to the set $\mathcal{S}_j \backslash \alpha_i$, makes it unavoidable set of patterns. To reach this, we proved that all proper subsets of the sets $\mathcal{S}_j, 1 \leq j \leq 13$ are avoidable sets of patterns. $\square$