Abelian Anti-Powers in Infinite Words

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Abstract

We introduce and study the notion of an abelian anti-power in the context of combinatorics on words. An abelian anti-power of order k (or simply an abelian k–anti-power) is a concatenation of k consecutive words of the same length having pairwise distinct Parikh vectors. This definition therefore generalizes to the abelian setting the notion of a k–anti-power, as introduced in [G. Fici et al., *Anti-powers in infinite words*, J. Comb. Theory, Ser. A, 2018], that is a concatenation of k pairwise distinct words of the same length. In particular, we deal with the question to determine whether a word contains abelian k–anti-powers for arbitrarily large k. A word with bounded abelian complexity clearly cannot contain abelian anti-powers of arbitrary order. We show that the Sierpiǹski word (whose abelian complexity grows logarithmically) does not contain abelian 11–anti-powers. Another question is to find words with low factor complexity that contain both abelian powers and abelian anti-powers of arbitrary order. We show that all paperfolding words have this property.

1 Introduction

Many of the classical definitions in combinatorics on words (e.g., period, run, power, factor complexity, etc.) have a counterpart in the abelian setting, though they may not enjoy the same properties.

Recall that the Parikh vector $P(w)$ of a word w over a finite ordered alphabet $\mathbb{A} = \{a_1, a_2, \ldots, a_{|\mathbb{A}|}\}$ is the vector whose i-th component is equal to the number of occurrences of the letter a_i in w, $1 \leq i \leq |\mathbb{A}|$. For example, the Parikh vector of $w = abbca$ over $\mathbb{A} = \{a, b, c\}$ is $P(w) = (2, 2, 1)$. This notion is at the basis of the abelian combinatorics on words, where two words are considered equivalent if and only if they have the same Parikh vector.

The fundamental result of Morse and Hedlund [4] (an infinite word is aperiodic if and only if its factor complexity is unbounded) does not hold anymore in the case of the abelian complexity (the function that counts the number of distinct Parikh vectors of factors of length n for each n), as there exist aperiodic words with bounded abelian complexity. In fact, Richomme et al. [5] have observed that if a word has bounded abelian complexity, then it contains abelian powers of any order — an abelian power of order k is a concatenation of k words having the same Parikh vector. However, this is not a characterization of words with bounded abelian complexity. Madill and Rampersad proved that the regular paperfolding word has unbounded abelian complexity [3], and Štěpán Holub proved that it contains abelian powers of any order [2].

In a recent paper [1], the first and the third author, together with Antonio Restivo and Luca Zamboni, introduced the notion of an anti-power. An *anti-power of order k*, or simply a *k–anti-power*, is a concatenation of k consecutive pairwise distinct words of the same length. E.g., *aabaaabbbaba* is a 4–anti-power.

In [1], it is proved that the existence of powers of any order or anti-powers of any order is an unavoidable regularity for infinite words:

Theorem 1. *[1] Every infinite word contains powers of any order or anti-powers of any order.*

Note that in the previous statement there is no hypothesis on the alphabet size.

In this paper, we extend the notion of an anti-power to the abelian setting.

Definition 2. An *abelian anti-power of order k*, or simply an *abelian k–anti-power*, is a concatenation of k consecutive words of the same length having pairwise distinct Parikh vectors.

For example, *aabaaabbbabb* is an abelian 4–anti-power. Notice that an abelian k–anti-power is a k–anti-power but the converse does not necessarily holds (which is dual to the fact that a k–power is an abelian k–power but the converse does not necessarily holds).

We think that an analogous of Theorem 1 may still hold in the case of abelian anti-powers, but unfortunately the proof of Theorem 1 does not seem to be generalizable to the abelian setting.

Problem 1. Does every infinite word contain abelian powers of any order or abelian anti-powers of any order?

Clearly, if a word has bounded abelian complexity, then it cannot contain abelian anti-powers of arbitrary order. However, we show in this paper that the converse is not true. Indeed, we prove that the Sierpiǹski word does not contain abelian 11–anti-powers. The Sierpiǹski word has logarithmic abelian complexity (by construction) and contains abelian powers of any order (since it contains arbitrarily long blocks of bs).

An infinite word can contain both abelian powers of any order and abelian anti-powers of any order. This is the case, for example, of any word with full factor complexity. However, finding a class of words with low factor complexity satisfying this property seems a more difficult task. Indeed, most of the well-known examples of aperiodic words (Thue-Morse, Sturmian words, etc.) have bounded abelian complexity, hence they cannot contain abelian anti-powers of any order — whereas, by the aforementioned remark of Richomme et al. [5], they contain abelian powers of any order. Building upon the theory that Štěpán Holub developed to prove that all paperfolding words contain abelian powers of any order [2], we prove that all paperfolding words contains also abelian anti-powers of any order.

References

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