Rigidity and Substitutive Dendric Words

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Dendric words are infinite words defined in terms of extension graphs that describe the left and right extensions of their factors. Extension graphs are bipartite graphs that can be roughly described as follows: given an infinite word x, and given a finite factor w of x, one puts an edge between left and right copies of letters a and b such that awb is a factor of x. Dendric words are defined by requiring that the extension graph of each of its factor is a tree. This class of words with linear factor complexity includes classical families of words such as Sturmian words, codings of interval exchanges, or else, Arnoux-Rauzy words. Dendric words have striking combinatorial, ergodic and algebraic properties. This includes in particular algebraic properties of their return words [4], and of maximal bifix codes defined with respect to their languages [2, 5, 6]. They have been introduced in [4] and studied in several papers (as, for instance, [5, 6]), under the name of tree words. We have chosen to call them here dendric words, and the subshifts they generate dendric subshifts, in order to avoid any ambiguity with respect to the notion of tree shift that refers to shifts defined on trees, and not on words (see e.g. [1]).

In this talk, based on [7], I investigate the properties of substitutive dendric words and prove some rigidity properties. Rigidity of an infinite word x has to do with the algebraic properties of its *stabilizer* $\mathrm{Stab}\,(x)$, that is the monoid of substitutions that fix it: an infinite word generated by a substitution is rigid if all the substitutions in $\mathrm{Stab}\,(x)$ are powers of a unique substitution. In this work, we concentrate on the iterative stabilizer according to the terminology of [9]: we focus on non-erasing morphisms and on infinite words generated by iterating a substitution.

There are numerous results on the two-letter case concerning rigidity (see [10, 11] and also [3]). It is indeed well known that Sturmian words generated by substitutions are rigid [10, 11]. The situation is more contrasted as soon as the size of the alphabet increases. For instance, over a ternary alphabet, the stabilizer of a given infinite word can be infinitely generated, even when the word is generated by iterating an invertible primitive morphism (see [8, 9]).

Our main results are the following, where §-adic expansions correspond to the limit of compositions of substitutions of the form $\sigma_1 \circ \cdots \circ \sigma_n$, $\S_e$ stands for the set of elementary positive automorphisms of the free group generated by the alphabet of the word and tame substitutions are elements of $\S_e^*$.

Theorem 1. *A recurrent dendric word over an alphabet A is primitive substitutive if and only if it has an eventually periodic primitive $\S_e$-adic representation.*

Theorem 2. *Let x be a dendric word. Primitive substitutions in the stabilizer Stab(x) of x coincide up to powers. More precisely, if x is a fixed point of both σ and τ primitive substitutions, then there exist $i, j \geq 1$ such that $\tau^i = \sigma^j$.*

Let x be a recurrent substitutive dendric word. There is a primitive tame substitution θ such that any primitive substitution $\sigma \in$ Stab(x) has a power that is (tamely) conjugate to a power of θ, that is, there exists a tame substitution τ such that $\sigma^i = \tau\theta^j\tau^{-1}$, for some $i, j \geq 1$.

In particular, if x is a dendric word, any primitive substitution in Stab(x) is a tame substitution.

Theorem 3. *Let (X, S) be an aperiodic minimal dendric subshift. Then it admits no rational topological eigenvalue.*

Corollary 4. *Let (X, S) be an aperiodic minimal dendric subshift. Then, it can neither be generated by a primitive constant length substitution, nor be a Toeplitz subshift.*

Our proofs rely on the notion of return words and on the so-called Return Theorem [4] that states that for every infinite dendric word defined over the alphabet A, the set of (right) return words is a basis of the free group generated by the alphabet A.

References

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