A definition and counting of biperiodic recurrent configurations in the sandpile model on $\mathbf{Z}^2$

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Abstract

For the sandpile model on the usual two dimensional grid, we propose a weaker version of Dhar criterion to define recurrent configurations among stable biperiodic configurations. We check this new criterion via an algorithm which auto-stabilises to a canonical ultimately periodic behaviour independent of details in its not fully specified initialisation. This leads to ultimately periodic edge/vertex traversals similar to those of Cori-Le Borgne [2] in the case of finite graphs and then to a bijection with *some* cycle-rooted forests on the torus describing the period. A determinantal formula [5] counts all those forests and the refinement with some monodromy parameters allows to identify in some coefficients the number of recurrent configurations.

The Abelian sandpile model was introduced by physicists Bak, Tang and Wiesenfeld in [1] as a model of self-organized criticality. Given a simple, undirected graph $(V \cup \{s\}, E)$ where we distinguish s as the sink of the graph, we consider *configurations* in this model which are an assignments $\eta : V \mapsto \mathbf{Z}$ of some grains of sand on each vertex. We say that η is stable at $x \in V$ if $\eta(x) < \deg(x)$, and η is stable if it is stable at all $x \in V$. If η is unstable at x, then x is allowed to topple which means that the vertex x sends one grain along each incident edge. This toppling is said *legal*. A toppling is *forced* when it is not necessarily legal. Grains arriving at the sink are lost. Given a configuration η, we define a stabilization as a sequence of allowed topplings until a stable configuration is reached. The result of all stabilizations is unique due to commutations of topplings of unstable vertices and is noted $stab(\eta)$.

Let $P(\eta)$ be the result of a stabilization of $\eta + \mathbf{1}_{s\sim}$, that is η with an extra grain on each neighbour of s, which may be interpreted as a forced toppling of the sink. $P(\eta)$ is also called the Dhar criterion since the set of recurrent configurations is a subset of the stable configurations characterized by Dhar [4] as the fixed points of P. For such a fixed point, each vertex topples exactly once in this process. The notion of recurrence is related to a natural Markov chain in this model not discussed here [3], and it is well studied for its connection with spanning trees [4], uniform spanning tree, the Tutte polynomial on the underlying graph [2]. Also on finite graphs, the set of recurrent configurations equipped with the operation $(\eta, \mu) \mapsto stab(\eta + \mu)$ is an abelian group [3]. When the graph is the grid $\mathbf{Z}^2$, the existence of such a group is open.

One of our motivations is the search of finite groups on a subset of recurrent configurations on $\mathbf{Z}^2$, which may be subgroups of the hypothetical (infinite) group. We focus on the subset of biperiodic configurations on the grid defined as follows. Let $\vec{P}_1, \vec{P}_2 \in \mathbf{Z}^2$ two non collinear vectors. A configuration η of $\mathbf{Z}^2$ is biperiodic of period $(\vec{P}_1, \vec{P}_2)$ if for all $x \in \mathbf{Z}^2$, $\eta(x+\vec{P}_1) = \eta(x+\vec{P}_2) = \eta(x)$.

Several approaches are suggested by litterature to define the notion of recurrence: for example, adding one edge per period to an extra vertex called the sink (dissipative sandpiles [6]), another example merges in one sink vertex all vertices outside a finite polygon and then scale this polygon [7]. Our approach relies on a weaker form of Dhar criterion and leads to a degenerate polygon which is an half-plane. Indeed we place the sink at infinity in a direction by analogy to the projective plane. The sink s is a point at infinity and will be describe by an euclidean vector $\vec{s} = (s_x, s_y) \in \mathbf{Z}^2$ where $gcd(s_x, s_y) = 1$. Note that $\vec{s}$ and $-\vec{s}$ refer to different sinks.

The sink being sent to infinity, the difficulty of toppling it appears. We replace this by a forced toppling of an half-plane of line boundary orthogonal to the sink $\vec{s}$.

Definition 1 (Weak Dhar criterion in a rational direction)**.** *A configuration* η *is said* recurrent in the direction of the sink $\vec{s}$ *if and only if for any* $k \in \mathbf{Z}$ *the forced toppling of the vertices of the half-plane* $\{(x, y) \in \mathbf{Z}^2 \mid s_x x + s_y y \geq k\}$ *leads to the legal toppling of all other vertices.*

Proposition 1. *There exists execution of the weak Dhar criterion on biperiodic configurations that is auto-stabilizing to an ultimately periodical behaviour which does not depend on the position of the half-plane defined by a line colinear to* $\vec{s}^{\perp}$ *and can be simulated in finite time.*

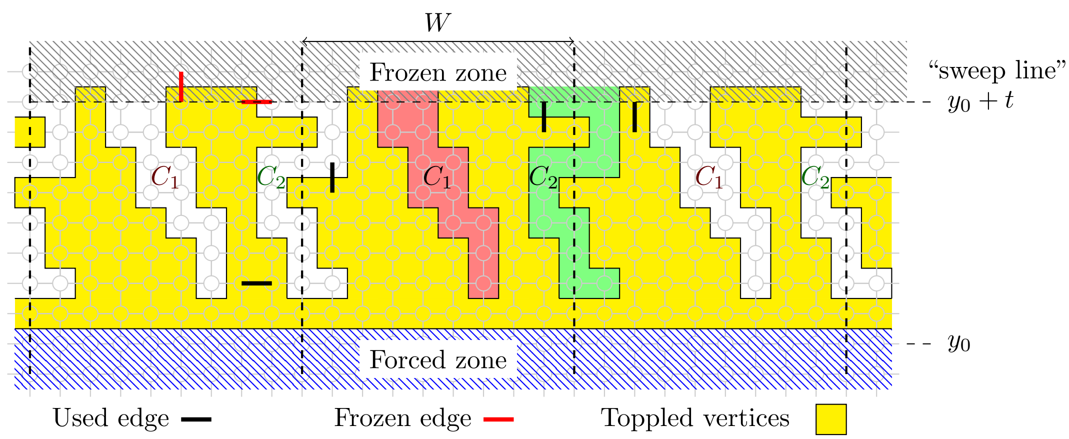

Figure 1: Weak Dhar criterion after step t, C_1 and C_2 are next connected components

Sketch of the proof when $\vec{s} = (0, -1)$ *(that can be generalized for all* $\vec{s}$*).* Let η be a recurrent configuration in direction $(0, -1)$ of period $(\vec{P}_1, \vec{P}_2)$. Without losing generality, we can assume that $\vec{P}_1 = (W, 0)$ and $\vec{P}_2 = (0, H)$ where $W, H > 0$. We equip the set of edges with the following order $\prec_{\vec{s}}$. Let e_1 (resp. e_2) be an edge of middle m_1 (resp. m_2) in the usual embedded of $\mathbf{Z}^2$, $e_1 \prec_{\vec{s}} e_2$ if and only if $\vec{s} \cdot \vec{m_1} < \vec{s} \cdot \vec{m_2}$ or ($\vec{s} \cdot \vec{m_1} = \vec{s} \cdot \vec{m_2}$ and $\vec{s}^{\perp} \cdot \vec{m_1} < \vec{s}^{\perp} \cdot \vec{m_2}$) where $\vec{s} \cdot \vec{m_1}$ is the usual scalar product between $\vec{s}$ and $\vec{m_1}$ and $\vec{s}^{\perp}$ is the vector $(-s_y, s_x)$. For $\prec_{(0,-1)}$, edges are ordered increasingly from top to bottom in priority, ties broken from left to right. When a vertex become unstable, it topples and its grains are sent along incident edges which become pending. Such a crossing grain is received at opposite endpoint when this pending edge is activated. We control the process of stabilisation by activating the maximal pending edge according to $\prec_{\vec{s}}$ (see [2] for details).

We start by a forced toppling of the half-plane $\vec{(x, y)} \cdot (-\vec{s}) = y \leq y_0 \in \mathbf{Z}$, denoted $H_{\leq y_0}$. The remaining legal topplings, more precisely edges allowed to be activated, are enclosed between this half-plane and a "sweep line" $y = y_0 + t$, where $t \geq 0$ is called a step of execution for Dhar criterion. More precisely at each step $t > 0$, we only consider edges which middle has ordinate $y \in [y_0, y_0 + t]$. Thus there is a forced toppled zone $y \leq y_0$, a frozen zone $y > y_0 + t$ and a working zone $y_0 < y \leq y_0 + t$, see Figure 1. Since η is recurrent, at step t there is at least one vertex v_t that topples on line $y = y_0 + t$. The order $\prec_{\vec{s}}$ guarantees that the set of connected components of untoppled vertices in the working zone is periodic of period $(W, 0)$ (C_1 and C_2 on Figure 1) and that these components are finite as long as η is recurrent since enclosed between the sweep line and the sequences of topplings leading to toppled vertices $(v_t + k\vec{P_1})_{k\in\mathbf{Z}}$. Thus the toppling of a vertex v is independent of the toppling of each $v + k(W, 0)$ with $k \in \mathbf{Z}^*$. This observation allows to simulate the criterion on a cylinder $[1, W] \times \mathbf{Z}$ with a periodic configuration of period $(0, H)$ for some H.

The more we force topplings, the more we topple vertices. This and translation symmetry $\vec{P_2}$ implies that if a vertex v topples at step $t + H$, then $v - (0, H) = v - \vec{P_2}$ topples no later than step t. As a corollary, at most one of the vertices $(v + k\vec{P_2})_{k\in\mathbf{Z}}$ can topple legally in any H consecutive steps. Thus for any H consecutive steps, there is at most WH vertices that topple.

We can show by contradiction that each vertex v topples at some step. For any vertex v, either all $(v + k\vec{P_2})_{k\in\mathbf{Z}}$ topples or there exists k_v such that exactly $(v + k\vec{P_2})_{k\geq k_v}$ do not topple. If not all vertices topples at some step, we can define $k_{WH} = \max_v k_v$ where v runs over the subset S of the vertices of $[1, W] \times [1, H]$ for which k_v is defined and $\overline{S} := [1, W] \times [1, H] \setminus S$ the complement subset. By definition, in the half-plane $H_{>y_0+k_{W,H}H}$, the subsets $(S + k\vec{P_2})_{k\geq k_{WH}}$ never topples and all other vertices in $(\overline{S} + k\vec{P_2})_{k\geq k_{W,H}}$ topples. From this, we also deduce the existence of a step $t_{W,H}$ such that no vertex in the half plane $H_{\leq y_0+k_{W,H}H}$ can legally topple at steps $t \geq t_{W,H}$. For steps $t \geq t_{W,H}$, only vertices in $(\overline{S} + k\vec{P_2})_{k\geq k_{W,H}}$ periodically topples. This describes an (infinite) sequence of legal topplings toward a stable configuration where $(S + k\vec{P_2})_{k\geq k_{WH}}$ did not topple, which is a contradiction with the recurrence of η (so $S = \emptyset$).

Hence, let T be the first step when all vertices of $[1, W] \times [1, H]$ has been toppled. Since there is at most WH vertices that topple at step T, the sequence of toppling that destabilizes the last untoppled vertex of $[1, W] \times [1, H]$ starts on line $y = y_0 + T$, is at most of length WH and ends to a line below $y = y_0 + H$ so $T \leq H + WH$. So the criterion is finite and effective.

Moreover, from steps $T + 1$ to $T + H$, we observe the ultimately periodic behaviour of the Dhar criterion: WH vertices topple, all having distinct copies in the fundamental domain $[1, W] \times [1, H]$. In addition, for $k \leq H$, the half-plane $H_{\leq y_0+k}$ has toppled at step T so forcing toppling of this half-plane, instead of $H_{\leq} y_0$, results after $T - k$ steps in the same set of toppled vertices. So the ultimately periodic behaviour of Dhar criterion starting either by $H_{\leq y_0}$ or $H_{\leq y_0+k}$ are the same. □

The proof induces a bijection with a subset of cycle rooted spanning forests [5] on the toroidal grid: from step $T + 1$ to step $T + H$, we attach to each vertex and its repetitions the edge that destabilizes it. In order to respect the order $\prec_{\vec{s}}$ on the cylinder, it is enough to process the toppling in only one repetition of each connected component at a given step. The result is a cycle rooted spanning forest of the toroidal grid with non contractible cycles. These cycles correspond to infinite periodic branches in the plane whose slopes are not orthogonal to $\vec{s}$.

Theorem 1. *Let $\vec{s}$ be a sink. The set of recurrent biperiodic configuration of pattern size $W \times H$ on $\mathbf{Z}^2$ is in bijection with the set of cycle rooted spanning forests of the toroidal grid $W \times H$ whose slope (a, b) is such that $a \cdot s_x + b \cdot s_y \neq 0$.*

We assume in the next part that $W, H \geq 2$. We call *NCRSF* a non-contractible cycle rooted spanning forests. We want to count the recurrent configurations in a direction $\vec{s}$. By definition of homology, the image of one copy of an (oriented) cycle of homology class $(i, j) \in \mathbf{Z}^2$ starting at (x, y) into the plane ends at $(x + jW, y + iH)$. Kenyon [5] gives the following determinantal formula $\sum_{\text{NCRSFs } \gamma} \left(2 - z^i w^j - z^{-i} w^{-j}\right)^k = \det \Delta$ where (i, j) is the homology class of the cycles of γ and k their number, z is the monodromy of cycles with homology class $(1, 0)$, w is the monodromy of cycles with homology class $(0, 1)$, Δ is the Laplacian on the line bundle with connection 1 over all

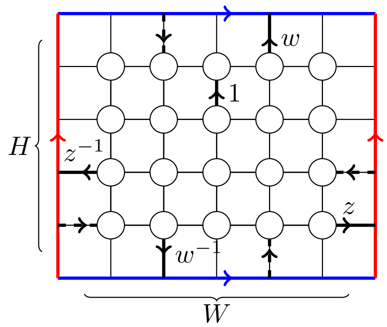

Figure 2: Connection Φ on the torus $W \times H$

kj \ ki	0	1	2	3	4
0		31300528	541732	1528	1
1	31300528	5427200	31232	4	
2	541732	31232	6		
3	1528	4			
4	1				

Figure 3: The number of NCRSF on a toroidal grid 4×4 in each direction where $k = gcd(ki, kj)$

oriented edges except those crossing the blue (connection w or w^{-1}) and red sides (connection z or z^{-1}) of a fix fundamental rectangle as in figure 2. Then $\Delta(v) = \sum_{u \to v} v - \Phi_{u \to v} u$ where $\Phi_{u\to v}$ is the connection value on edge $u \to v$.

Due to planarity of the grid, a cycle cannot cross itself on the toroidal grid. Thus the homology class (i, j) of a cycle in this graph has $gcd(i, j) = 1$. Moreover the length of such a cycle is at least $|Wj| + |Hi|$. That gives the first part of the following proposition.

Proposition 2. *Given γ a NCRSF on a toroidal grid $W \times H$, if γ has k cycles with homology class (i, j) then i and j are co-prime and $|kjW| + |kiH| \leq WH$. Reciprocally for any $(i, j) \in \mathbf{Z}^2$ co-prime and any $k > 0$ such that $|kjW| + |kiH| \leq WH$, there exists a NCRSF of parameters (i, j, k).*

The second part of this proposition is achieved by k repetitions of a digital line from Bresenham's line algorithm (with corners).

We denote $Q_{i,j,k} = \left(2 - z^i w^j - z^{-i} w^{-j}\right)^k$ for any $k > 0$ and any $(i, j) \in S = \{(0, 1)\} \cup \{(a, b) \mid a > 0 \text{ and } gcd(a, b) = 1\}$, the $(Q_{i,j,k})_{i,j,k}$ are linearly independent. So the number of NCRSFs is the sum of the coefficients $(\alpha_{i,j,k})_{i,j,k}$ of $\det \Delta$ in the decomposition in $(Q_{i,j,k})_{i,j,k}$: $\det \Delta = \sum_{i,j,k} \alpha_{i,j,k} Q_{i,j,k}$. One can show that $\alpha_{i,j,k} = \alpha_{i,-j,k}$ for $i > 0$.

Proposition 3. *The number of biperiodic recurrent configurations in direction $\vec{s}$ of size $W \times H$ is $\sum_{(i,j,k) \in S_{\vec{s}}} \alpha_{i,j,k}$ where $S_{\vec{s}} = \{(i, j, k) \mid (i, j) \in S, k > 0, |kjW| + |kiH| \leq WH, iWs_x + jHs_y \neq 0\}$.*

This formula enhances the counting the recurrent configurations in a direction that was limited to enumeration. Some explicit results are given on Figure 3. Some extra results up to $m \times n$ with $m, n \leq 9$ can be found at `https://www.labri.fr/perso/hderycke/biperiodic_recurrent/`.

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