Generalized Beatty sequences and complementary triples

Abstract

A Beatty sequence is the sequence $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$, where α is a positive real number. What Beatty observed is that when B is the sequence $B(n) = \lfloor n\beta \rfloor$, with α and β satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{1}$$

then $(A(n))$ and $(B(n))$ are *complementary* sequences, that is, the sets $\{A(n) : n \geq 1\}$ and $\{B(n) : n \geq 1\}$ are disjoint and their union is the set of positive integers.

A *generalized Beatty sequence* is a sequence v defined by $v(n) = p\lfloor n\alpha \rfloor + qn + r$, where α, p, q, r are real numbers. These occur in several problems, as for instance in homomorphic embeddings of Sturmian languages in the integers ([1]).

Question 1 Let α be an irrational number, and let A defined by $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$ be the Beatty sequence of α. Let Id defined by $\mathrm{Id}(n) = n$ be the identity map on the integers. For which sixtuples of integers p, q, r, s, t, u are the two sequences

$$v = pA + q\,\mathrm{Id} + r \quad \text{and} \quad w = sA + t\,\mathrm{Id} + u$$

complementary sequences?

Question 2 For which nonatuples of integers $(p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3)$ the three sequences

$$v_i = p_i A + q_i\,\mathrm{Id} + r_i, \; i = 1, 2, 3$$

are a complementary triple?

Here a *complementary triple* are three sequences, with the property that the sets they determine are disjoint with union the positive integers.

In this talk, based on joint work with Jean-Paul Allouche I give (incomplete) answers to these questions.

[1] Michel Dekking, The Frobenius problem for homomorphic embeddings of languages into the integers, Theoretical Computer Science 732(2018),73-79.