ON THE GROUP OF A RATIONAL MAXIMAL BIFIX CODE

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Abstract. We give necessary and sufficient conditions for the group of a rational maximal bifix code Z to be isomorphic with the F-group of $Z \cap F$, when F is recurrent and $Z \cap F$ is rational. The case where F is uniformly recurrent receives special attention.

1. Introduction

In the past few years, special attention has been given to bifix codes which may not be maximal but are maximal within some language, which is usually chosen to be recurrent or uniformly recurrent. This line of research has produced new and strong connections between bifix codes, subgroups of free groups and symbolic dynamical systems (cf. [4] and the sequels [5, 6, 7, 8]).

If Z is a thin maximal bifix code and F is a recurrent set, then $X = Z \cap F$ is an F-maximal bifix code, that is, a maximal bifix code within F (with X finite if F is uniformly recurrent) [4]. This leads to a process of "relativization" of several previously known definitions for maximal codes. An important example is the group $G(Z)$ of a rational code Z, i.e., the Schützenberger group of the minimum ideal $J(Z)$ of the syntactic monoid $M(Z^*)$ of Z^*. In this case, the relativization consists in taking the intersection $X = Z \cap F$ and the Schützenberger group of the minimum $\mathcal{J}$-class $J_F(X)$ that intersects the image of F in $M(X^*)$, when X is rational. This group, denoted by $G_F(X)$, is the *F-group of X*. How are $G(Z)$ and $G_F(X)$ related? They are not always isomorphic, even if Z is a group code (i.e., Z is a code with $M(Z^*)$ a finite group) and F is uniformly recurrent. In [4] it is shown that if Z is a group code and F is Sturmian, then $G(Z)$ and $G_F(X)$ are isomorphic. This was extended to tree sets in the manuscript [10], thanks to a novel approach consisting in exploring links between $G(Z)$, $G_F(X)$ and the Schützenberger (profinite) group $G(F)$ of the minimum $\mathcal{J}$-class $J(F)$ of the topological closure of F within the free profinite monoid generated by the alphabet of F, and, with the help of these links, taking advantage of results on $G(F)$ from [2, 3]. Building on this approach, we get new results about when $G(Z) \simeq G_F(X)$ holds, recovering previous results in the process.

2. Preliminaries on free profinite monoids

Here A is always a finite alphabet. Take $u, v \in A^*$. If $u \neq v$, there is a homomorphism $\varphi\colon A^* \to M$ onto a finite monoid such that $\varphi(u) \neq \varphi(v)$.

2010 *Mathematics Subject Classification.* 20M05, 20E18, 37B10, 68R15.

Key words and phrases. maximal bifix code, rational code, group code, syntactic monoid, F-group, Schützenberger group, uniformly recurrent set, free profinite monoid.

Let $r(u,v)$ be the minimum for $|M|$. We may consider the metric d on A^* such that $d(u,v) = 2^{-r(u,v)}$ if $u \neq v$. The *free profinite monoid* $\widehat{A^*}$ is the compact monoid resulting from the completion of A^* under d, a terminology justified as $\widehat{A^*}$ is the free object in the class of A-generated profinite monoids. For an extended introduction, see [11]. The elements of $\widehat{A^*}$ are called *pseudowords* over A. Words of A^* are topologically isolated in $\widehat{A^*}$. Pseudowords generalize words, but the structure of $\widehat{A^*}$ is richer than that of A^*. Next is a glimpse of that. If F is a factorial subset of A^*, then the topological closure $\overline{F}$ is itself factorial in $\widehat{A^*}$, and when F satisfies the stronger property of being recurrent, there is a minimum $\mathcal{J}$-class $J(F)$ contained in $\overline{F}$, which moreover is regular. Maximal subgroups of $J(F)$ were identified in [2, 3]. We mention that the proper factors (that is, strictly $\mathcal{J}$-above) of $u \in \widehat{A^*} \setminus A^*$ belong to A^* if and only if $u \in J(F)$ for some uniformly recurrent set F [1].

3. Preparatory technical results

In this section we prepare the main results of the next section.

Recall that a *parse* of a word w with respect to a subset X of A^* is a triple (v, x, u) such that $w = vxu$ with $v \in A^* \setminus A^*X$, $x \in X^*$ and $u \in A^* \setminus XA^*$. The number of parses of w with respect to X is denoted by $\delta_X(w)$. The *F-degree* of X is $d_F(X) = \sup\{\delta_X(w) \mid w \in F\}$. The *degree* of X is $d(X) = d_{A^*}(X)$. For F recurrent containing a bifix code X, one has $d_F(X)$ finite if and only if X is F-thin and an F-maximal bifix code [4].

The notion of parse was generalized to pseudowords in [10], and so we may extend δ_X to $\widehat{A^*}$. Since then, we obtained the following useful tool.

Proposition 3.1. *Consider a factorial set F of A^*. Let X be a rational subset of F with finite F-degree d. Then $\delta_X(w) \leq d$ for every $w \in \overline{F}$, and the mapping $\delta_X \colon \overline{F} \to \{1, \ldots, d\}$ thus defined is continuous, if we endow $\{1, \ldots, d\}$ with the discrete topology.*

A pseudoword u is *forbidden in $Y \subseteq \widehat{A^*}$* if u is not a factor of an element of Y. The next proposition was deduced with the help of Proposition 3.1. We explain the notation used there. Let L be a rational language of A^*. By the universal property of $\widehat{A^*}$, the syntactic homomorphism $\eta_L \colon A^* \to M(L)$ admits a unique extension to a continuous homomorphism $\hat{\eta}_L \colon \widehat{A^*} \to M(L)$.

Proposition 3.2. *Let Z be a rational maximal bifix code of A^*. Suppose that F is a recurrent subset of A^* and that the intersection $X = Z \cap F$ is rational. The equality $d_F(X) = d(Z)$ holds if and only if the elements of $J(F)$ are forbidden in $\overline{Z}$. Moreover, if $d_F(X) = d(Z)$ then $\hat{\eta}_{Z^*}(J(F)) \subseteq J(Z)$.*

Consider a language L of A^*. Let $u, v \in A^*$. By definition, $\eta_L(u) \leq \eta_L(v)$ if and only if the context of u is contained in the context of v.

Proposition 3.3. *Let Z and F be subsets of A^*, with F factorial, and let $X = Z \cap F$. Suppose Z^* and X^* are rational. Let $e, f \in \widehat{A^*} \setminus \{1\}$ be idempotents, and let $u, v \in \widehat{A^*}$ with $u = euf$, $v = evf$ and $u \in \overline{F}$. Then:*

(1) $\hat{\eta}_{X^*}(u) \leq \hat{\eta}_{X^*}(v) \Rightarrow \hat{\eta}_{Z^*}(u) \leq \hat{\eta}_{Z^*}(v)$, if e and f are forbidden in $\overline{Z}$;

(2) $\hat{\eta}_{Z^*}(v) \leq \hat{\eta}_{Z^*}(u) \Rightarrow \hat{\eta}_{X^*}(v) \leq \hat{\eta}_{X^*}(u)$, if e and f are forbidden in $\overline{X}$.

Applying the preceeding tools, we deduce relationships between the maximal subgroups of $J(F)$, $J(Z)$ and $J_F(Z \cap F)$, for suitable Z and F.

Theorem 3.4. *Let F be a factorial subset of A^*, take a rational prefix code Z of A^*, and suppose $X = Z \cap F$ is a rational F-maximal prefix code. Let H be a maximal subgroup of $\widehat{A^*}$ with $H \subseteq \overline{F}$ and the elements of H being forbidden in $\overline{Z}$. Consider the maximal subgroup H_X of $M(X^*)$ containing $\hat{\eta}_{X^*}(H)$ and the maximal subgroup H_Z of $M(Z^*)$ containing $\hat{\eta}_{Z^*}(H)$. There is an injective homomorphism $\alpha\colon H_X \to H_Z$ such that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\hat{\eta}_{Z^*}} & H_Z \\ {\scriptstyle \hat{\eta}_{X^*}}\downarrow & \nearrow{\scriptstyle \alpha} & \\ H_X & & \end{array} \tag{3.1}$$

commutes.

4. Main results

Based on the technical results of the previous section, namely Theorem 3.4, we deduce our main results.

Let F be a recurrent set of A^*. Say that a rational code Z of A^* is F-*charged* if $\hat{\eta}_{Z^*}$ maps maximal subgroups of $J(F)$ onto maximal subgroups of $J(Z)$. A rational code X contained in F is *weakly F-charged* if $\hat{\eta}_{X^*}$ maps maximal subgroups of $J(F)$ onto maximal subgroups of $J_F(X)$.

Theorem 4.1. *Consider a recurrent subset F of A^* and a rational bifix code Z of A^* with finite degree such that $X = Z \cap F$ is rational. Let H be a maximal subgroup of $J(F)$. The following conditions are equivalent:*

1. *Z is F-charged;*
2. *$d_F(X) = d(Z)$, $G_F(X) \simeq G(Z)$ and X is weakly F-charged;*
3. *$d_F(X) = d(Z)$, $|G_F(X)| = |G(Z)|$ and X is weakly F-charged.*

Recall that if $F \subseteq A^*$ is (uniformly) recurrent and Z is a maximal bifix code of A^*, then $Z \cap F$ is an F-maximal bifix (finite) code [4].

We show that a group code of A^* is F-charged when F is an uniformly recurrent *connected* set (that is, with only connected extension graphs, see [5]) with alphabet A. Therefore, we get the following corollary.

Corollary 4.2. *If Z is a group code of A^* and F is a uniformly recurrent connected set of alphabet A, then $d(Z) = d_F(Z \cap F)$ and $G(Z) \simeq G_F(Z \cap F)$.*

We say that a rational code Z is *nil-simple* if all idempotents of $M(Z^*)$ are in $J(Z)$. Group codes and finite codes are nil-simple. If F is uniformly recurrent and Z is nil-simple, the equality $d_F(X) = d(Z)$ in Theorem 4.1 becomes redundant, as seen next.

Theorem 4.3. *Let Z be a uniformly recurrent subset of A^*, and let Z be a nil-simple rational maximal bifix code Z of A^*. The following are equivalent:*

1. *Z is F-charged;*
2. *$G_F(Z \cap F) \simeq G(Z)$ and $Z \cap F$ is weakly F-charged;*
3. *$|G_F(Z \cap F)| = |G(Z)|$ and $Z \cap F$ is weakly F-charged.*

Moreover, if Z is F-charged, then $d(Z) = d_F(Z \cap F)$.

The special case of Corollary 4.2 in which Z is a group code and F is Sturmian was first proved in [4].

We also studied F-groups as permutation groups acting in a natural manner. In what follows, Q_Y is the set of vertices of the minimal automaton of Y^*, and i_Y is the corresponding initial state.

Theorem 4.4. *Let F be a recurrent subset of A^*. Suppose that Z is a rational bifix code of finite degree d. Let $X = Z \cap F$ and suppose that X is rational. Let H be a maximal subgroup of $J(F)$ such that $H_Z = \hat{\eta}_{Z^*}(H)$ is a maximal subgroup of $J(Z)$, and let $H_X = \hat{\eta}_{X^*}(H)$. Take the map $f\colon Q_X \cdot H_X \to Q_Z \cdot H_Z$ given by $f(i_X \cdot u) = i_Z \cdot u$, for $u \in H$, and take the unique group isomorphism $\alpha\colon H_X \to H_Z$ such that Diagram (3.1) commutes. Then the pair (f, α) is an equivalence of permutation groups with degree d.*

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