# Number of valid decompositions of Fibonacci prefixes

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**Abstract**

We establish several recurrence relations and an explicit formula for the number of factorizations of the length-$n$ prefix of the Fibonacci word into a (not strictly) decreasing sequence of standard Fibonacci words (OEIS sequence A300066).

## 1 Introduction

Extended Ostrowski numeration systems were introduced in [4] to solve a problem on palindromes in Sturmian words. A representation of $n$ in such a system related to a given Sturmian slope corresponds to a factorization of the prefix of length $n$ of the standard Sturmian word of this slope as a concatenation of finite standard words in a non-strictly decreasing order. Since in this abstract we consider only the Fibonacci case, it is reasonable to give a Fibonacci example: consider the prefix *abaababaabaaba* of the Fibonacci word of length 14 and its decompositions to standard words $s_0 = a$, $s_1 = ab$, $s_2 = aba$, $s_3 = abaab$, $s_4 = abaababa$, $s_5 = abaababaabaab$ in a decreasing order. We see that

$$\begin{aligned}
abaababaabaaba &= (abaababaabaab)(a) = s_5 s_0 \\
&= (abaababa)(abaab)(a) = s_4 s_3 s_0 \\
&= (abaababa)(aba)(ab)(a) = s_4 s_2 s_1 s_0 \\
&= (abaababa)(aba)(aba) = s_4 s_2 s_2 \\
&= (abaab)(aba)(aba)(aba) = s_3 s_2 s_2 s_2 \\
&= (abaab)(aba)(aba)(ab)(a) = s_3 s_2 s_2 s_1 s_0.
\end{aligned}$$

These six factorizations correspond to six *valid* representations of 14:

$$14 = \overline{100001} = \overline{11001} = \overline{10111} = \overline{10200} = \overline{1300} = \overline{1211}.$$

If we restrict ourselves to representations corresponding to strictly decreasing sequences, or, which is the same, to the representations only containing zeros and ones, their number for each $n$ is equal to the well-studied OEIS sequence A000119 (see, e.g., [2]). In particular, the lower limit of the sequence is 1, and the upper asymptotics grows as $O(\sqrt{n})$. But here we consider the number of all valid representations of $n$, denoted by $T(n)$, so that, for example, $T(14) = 6$, the obtained sequence is new and was just recently uploaded to the OEIS as A300066. Here we prove a series of recurrence relations and an explicit formula for it.

# 2 Result

Let $\varphi$ denote the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$. The Fibonacci word is a Sturmian word $s = s[1]s[2]\cdots$ of the slope $1/(\varphi+1) = 1/\varphi^2$ and of zero intercept, that is, for all $n$, we have

$$s[n] = \begin{cases} a, & \text{if } \{n/\varphi^2\} < 1 - 1/\varphi^2, \\ b, & \text{otherwise.} \end{cases} \tag{1}$$

Here $\{x\}$ denotes the fractional part of $x$. Another way to construct $s$ is to consider it as a limit $s = \lim s_n$ of finite standard words

$$s_{-1} = b, s_0 = a, s_{n+1} = s_n s_{n-1} \text{ for all } n \geq 0. \tag{2}$$

We write $N = \overline{k_n \cdots k_0}$ and call this representation of $N$ *valid* if $k_i \geq 0$ for all $i$ and $s(0..N] = s_n^{k_n} s_{n-1}^{k_{n-1}} \cdots s_0^{k_0}$, where $s(0..N]$ is the prefix of length $N$ of the Fibonacci word. The number of valid representations of $N$ is denoted by $T(N)$.

**Proposition 1.** *If $s[n] = a$, all valid representations of $n$ end with an even number of 0s. If $s[n] = b$, all of them end with an odd number of 0s.*

The main result of this abstract is the following

**Theorem 1.** *If $s[n] = a$, then $T(n) = \lceil n/\varphi^2 \rceil$, or, which is the same, $T(n)$ is equal to the number of occurrences of $b$ to $s(0..n]$ plus one. If $s[n] = b$, then $T(n) = \lceil n/\varphi^3 \rceil$, or, which is the same, $T(n)$ is equal to the number of occurrences of $aa$ to $s(0..n]$ plus one.*

The proof of the theorem is based on several recurrence relations on $T(n)$:

**Proposition 2.** *For all $\overline{s}$, $T(\overline{r0}) \geq T(\overline{r})$. If $r = r'10^{2k}$ for some $k \geq 0$, then $T(\overline{r0}) = T(\overline{r})$.*

**Proposition 3.** *For all $z \in \{0,1\}^*$ and for all $k \geq 1$,*

$$T(\overline{z10^{2k}}) = T(\overline{z10^{2k-2}}) + T(\overline{z(01)^k}).$$

**Proposition 4.** *For all $z \in \{0,1\}^*$ and for all $k \geq 1$,*

$$T(\overline{z10^k1}) = \begin{cases} T(\overline{z10^{k+1}}), & \text{if } k \text{ is odd,} \\ T(\overline{z10^k}) + T(\overline{z(01)^{k/2}}), & \text{if } k \text{ is even.} \end{cases}$$

Propositions 2 to 4 give a full list of recurrence relations sufficient to compute $T(n)$ for every $n > 1$, starting from $T(1) = 1$. In particular, as corollaries, we get simple formulas on the values of $T$ on Fibonacci numbers and their predecessors: starting from $F_1 = 1$, $F_2 = 2$, $F_{n+2} = F_{n+1} + F_n$, we get

$$T(F_{2n-1}) = T(F_{2n}) = F_{2n-3} + 1$$

and

$$T(F_{2n} - 1) = T(F_{2n+1} - 1) = F_{2n-2}.$$

The same recurrence relations serve to prove Theorem 1.

## 3 Acknowledgement

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## References

[1] J.-P. Allouche, J. Shallit. Automatic Sequences: Theory, Applications, Generalizations. Cambridge University Press, 2003.

[2] J. Berstel. An exercise on Fibonacci representations. Theor. Inform. Appl. 35 (2002) 491–498.

[3] J. Berstel, P. Séébold. Sturmian words. In: M. Lothaire, Algebraic combinatorics on words. Cambridge University Press, 2002. Chapter 2, pp. 45–110.

[4] A. E. Frid, Sturmian numeration systems and decompositions to palindromes. European J. Combin. 71 (2018) 202–212.

[5] C. G. Lekkerkerker. Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci. Simon Stevin 29 (1952) 190–195.

[6] A. Ostrowski. Bemerkungen zur Theorie der diophantischen Approximationen. Hamb. Abh. 1 (1921) 77–98.

[7] E. Zeckendorf. Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. Bull. Soc. R. Sci. Liège 41 (1972) 179–182.